Introduction to Ergodic Control

ICRA 2024: Ergodic Control Tutorial Workshop

Katie Fitzsimons

Mechanical Engineering
Pennsylvania State University

ICRA, May 2024

Definitions of Ergodicity

- **Markov Chains**
- **Signal Analysis**
- **Dynamic Systems**

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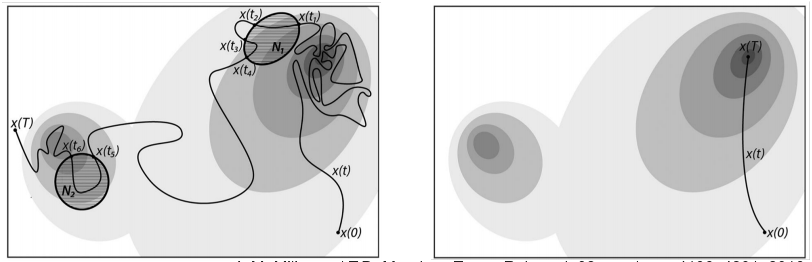
In this workshop, we will be referring to the ergodicity of a trajectory $x(t)$ with respect to a distribution $\phi(x)$

What makes a trajectory ergodic?

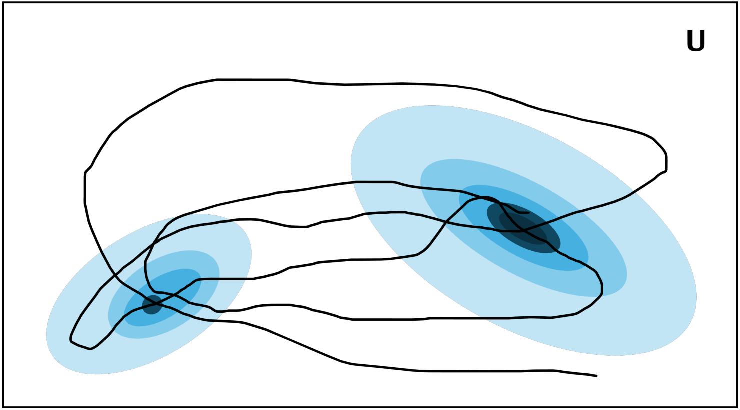
A trajectory is perfectly ergodic with respect to a distribution if the amount of time spent a neighborhood $\mathcal{N}$ of the state space is proportional to the spatial distribution in the neighborhood $\int N\phi(s)ds$.



L.M. Miller and T.D. Murphey, Trans. Rob., vol. 32, no. 1, pp. 4196–4201, 2016

Quantifying Ergodic trajectories

Given a trajectory and distribution in the rectangular domain U



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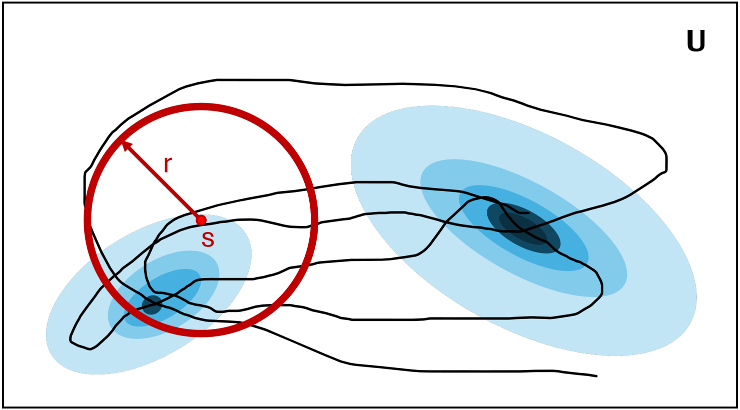
Define a set $B(s,r)$ and $I_{(s,r)}(y) = \begin{cases} 1 & inside\ B(s,r) \\ 0 & o.w. \end{cases}$



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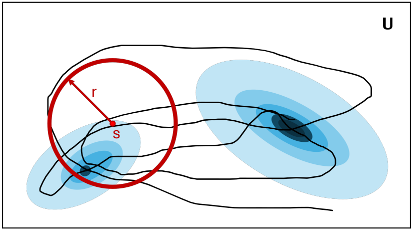

Define a set $B(s,r)$ and

$$I_{(s,r)}(y) = \begin{cases} 1 & inside\ B(s,r) \\ 0 & o.w. \end{cases}$$

The average time spent in the set $B(s,r)$ is

$$d^t(s,r) = \frac{1}{t}\int_0^t I_{(s,r)}(x(\tau))d\tau.$$

The measure of the distribution on the same set is given by

$$\bar{\phi}(s,r) = \int_U \phi(y) I_{(s,r)}(y)dy.$$

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This **metric** tells us the **distance from ergodicity**

Construct a trajectory distribution

Let's consider instead an alternative way to represent the trajectory, by constructing a distribution,

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*So we could rewrite average time spent in $B(s,r)$ as

$$d^t(s,r) = \langle C, I_{(s,r)}\rangle$$

.

Construct a trajectory distribution

Let's write the Fourier coefficients of $x(t)$ as

$$c_k = \langle C, F_k \rangle = \frac{1}{T} \int_0^T F_k(x(t))dt,$$

using Fourier basis functions of the form,

$$F_k(x(t)) = \frac{1}{h_k} \prod_{i=1}^{n} cos \left(\frac{k_i \pi}{L_i} x_i(t) \right).$$

- $x(t)$ is n-dimensional
- k is multi-index over the coefficients of the multi-dimensional Fourier transform
- L_i is a measure of the length of the dimension.
- h_k is a normalizing factor that make F_k an orthonormal basis

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In 2 dimensions, h_k takes the form,

$$h_k = \left(\int_0^{L_1} \int_0^{L_2} cos^2(\frac{k_1 \pi}{L_1} x_1) cos^2(\frac{k_2 \pi}{L_2} x_2) dx_1 dx_2\right)^{1/2}.$$

Defining a metric

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Now, the Fourier representations of C and ϕ are in the same vector space.
The distance between C and $\phi(x)$ is

$$\varepsilon(t) = \sum_{k_1=0}^{K} ... \sum_{k_n=0}^{K} \Lambda_k |c_k - \phi_k|^2$$

The coefficient $\Lambda_k = (1 + ||k||^2)^{-s}$ where $s = \frac{n+1}{2}$ places larger weights on lower frequency information.

Defining a metric

Requiring

$$\lim_{t \to \infty} \varepsilon = 0$$

is equivalent to requiring the time average of the **trajectory** to converge to the spatial averages of **distribution**.

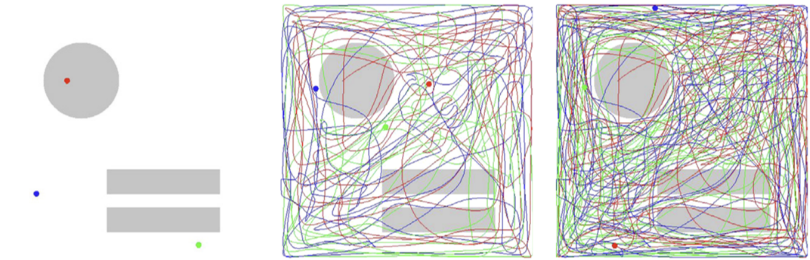
It can be shown that $\varepsilon(t)$ and $E(t)$ are equivalent metrics, since there exists bounded constants such that

$$C_1 \varepsilon(t) \leq E^2(t) \leq C_2 \varepsilon(t).$$

Spectral Multiscale Coverage

If the aim is to control agents with linear dynamics to cover a region uniformly, one can design a feedback law that drives the agents to take actions that maximize the rate of decay of the ergodic metric.

G. Mathew, I. Mezić / Physica D 240 (2011) 432–442



(a) Time, $t = 0.0$. (b) Time, $t = 6.0$. (c) Time, $t = 12.0$.

Ergodic Metric as a cost function

Given a system with dynamics,

$$\dot{x} = f(x(t), u(t)), \ x(0) = x_0.$$

Instead of

$$J = \int_0^t l(x,u)dt = \int_0^t (x_d - x)^T Q(x_d - x) + u^T Ru\, dt,$$

We use the ergodic metric

$$\begin{aligned} J(x(t), u(t)) &= q\,\varepsilon(x(t)) + \int_0^T u(t)^T Ru(t)dt \\ &= q \sum_{k_1=0}^{K} \dots \sum_{k_n=0}^{K} \Lambda_k \left(\frac{1}{T} \int_0^T F_k(x(t))dt - \phi_k \right)^2 \\ &\quad + \int_0^T u(t)^T Ru(t)dt. \end{aligned}$$

Derivative of an Ergodic Objective

We can find the gradient by taking the directional derivative of:

$$\frac{d}{d\epsilon}J(\xi+\epsilon\zeta)|_{\epsilon=0} =$$

$$\frac{d}{d\epsilon}\left[q\sum_{k=0}^{K}\Lambda_k\left(\frac{1}{T}\int_0^T F_k(x(s)+\epsilon z(s))dt-\phi_k\right)^2\right.$$

$$\left.+\int_0^T (u(t)+\epsilon v(t))^T R(u(t+\epsilon v(t))dt\right]_{\epsilon=0}$$

Derivative of an Ergodic Objective

Let's focus on the ergodic term for now

$$\frac{d}{d\epsilon} q\, \varepsilon(x(t) + \epsilon z(t))|_{\epsilon=0} =$$

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$$\frac{d}{d\epsilon} q\,\varepsilon(x(t)+\epsilon z(t))|_{\epsilon=0} =$$
$$\left[q \sum_{k=0}^{K} \Lambda_k \left[2\left(\frac{1}{T}\int_0^T F_k(x(s)+\epsilon z(s))ds - \phi_k \right) \cdot \int_0^T \frac{1}{T} DF_k(x(t)+\epsilon z(t))z(t)dt \right]\right]_{\epsilon=0}$$

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& \left. \cdot \frac{1}{T} DF_k(x(t)) \right] \cdot z(t) + R(t)u(t) \cdot v(t) dt
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When this derivative is close to zero, we have found a local extrema, and we can set up an LQ problem.

Summary of the Spectral Ergodic Metric

Write the Fourier coefficients of $x(t)$ as

$$c_k = \langle C, F_k \rangle = \frac{1}{T} \int_0^T F_k(x(t))dt,$$

Using the same basis functions, we can compute the coefficients of the spatial distribution,

$$\phi_k = \langle \phi(x), F_k \rangle = \int_X \phi(x) F_k(x) dx$$

So the distance from perfect ergodicity is

$$\varepsilon(t) = \sum_{k_1=0}^{K} \ldots \sum_{k_n=0}^{K} \Lambda_k |c_k - \phi_k|^2$$