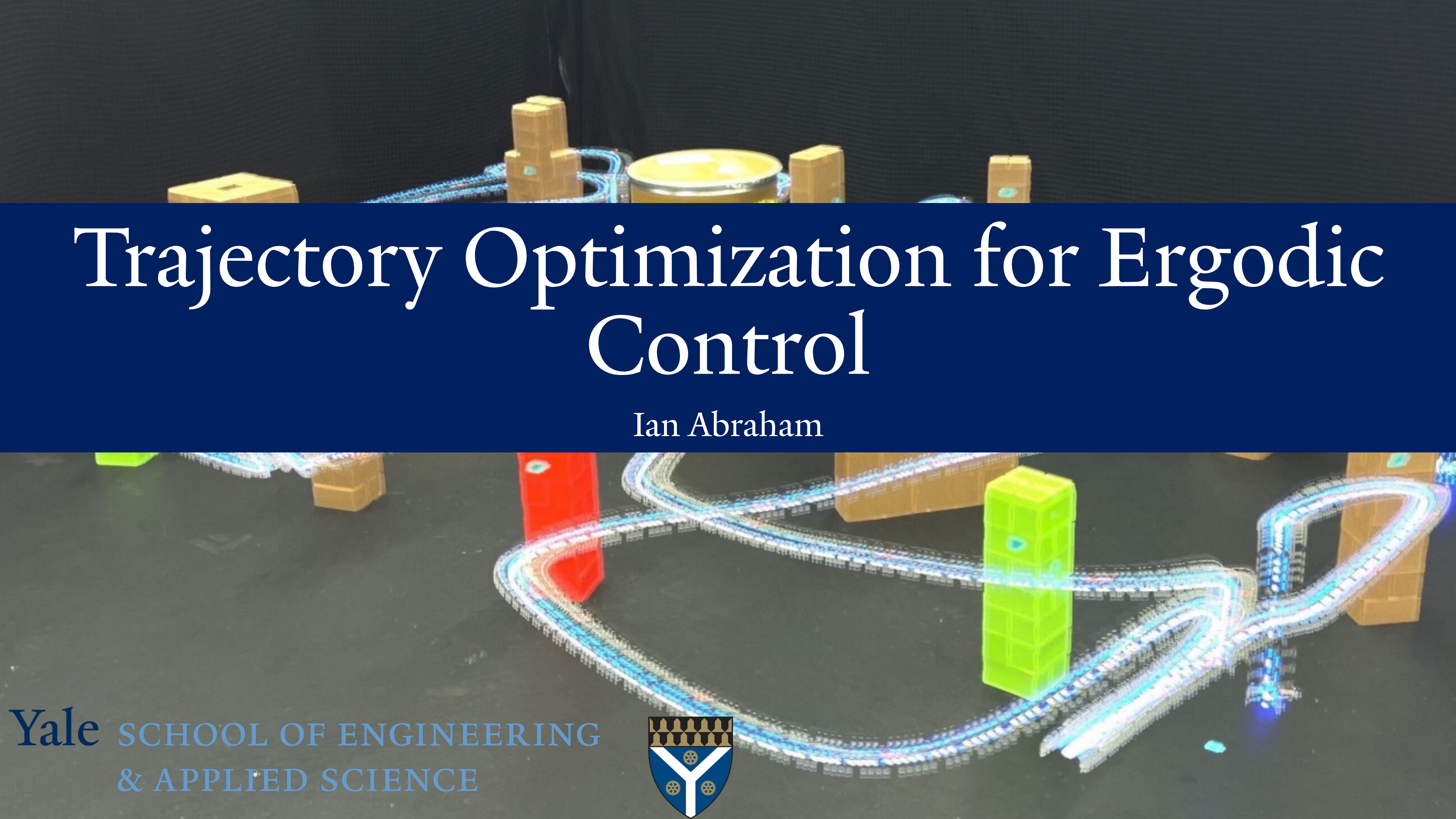
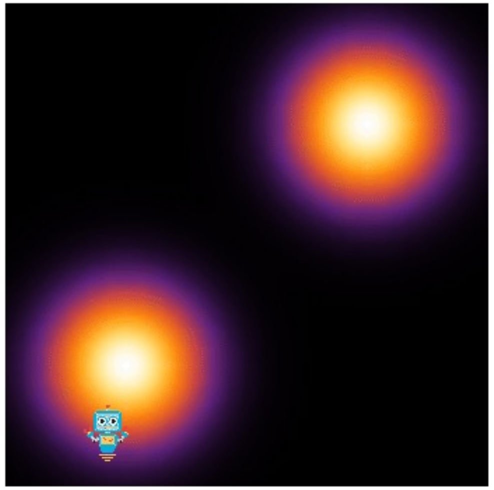


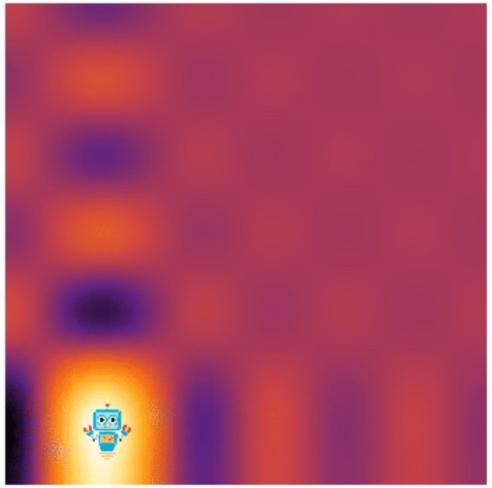
Trajectory Optimization for Ergodic Control

Ian Abraham

Yale SCHOOL OF ENGINEERING & APPLIED SCIENCE

Introduction to Ergodic Coverage





Ergodic Exploration/Search

- Goal: achieve ergodic dynamics over bounded area
- Approach: optimize ergodic metric over trajectories
  - Nonlinear/nonconvex problem
  - Many "good" local minima
  - Extendable to arbitrary constraints
- Independent of robot dynamics/spatial scale/sensor/information measure
- Non-Myopic Exploration (formulated over long planning horizons)

[1] Miller, Lauren M., Yonatan Silverman, Malcolm A. MacIver, and Todd D. Murphey. "Ergodic exploration of distributed information." IEEE Transactions on Robotics 32, no. 1 (2015): 36-52.

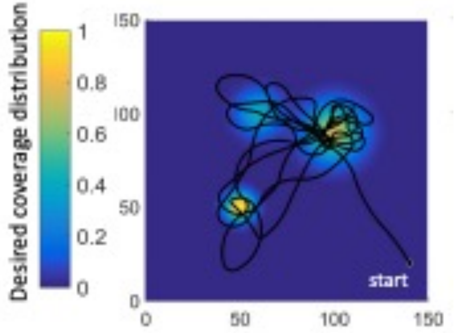
[2] Lerch, Cameron, Dayi Dong, and Ian Abraham. "Safety-critical ergodic exploration in cluttered environments via control barrier functions." In 2023 IEEE International Conference on Robotics and Automation (ICRA), pp. 10205-10211. IEEE, 2023.

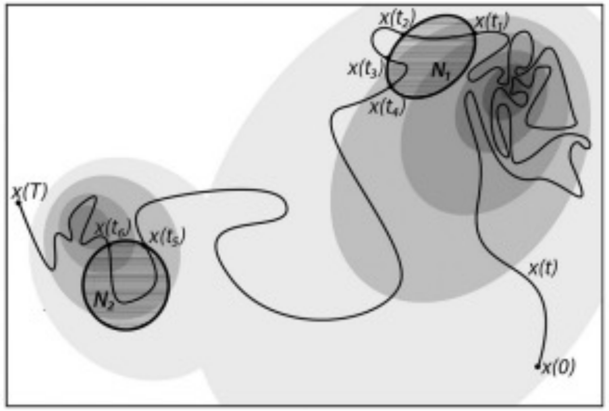
Introduction to Ergodic Coverage

Ergodic Coverage

- A trajectory is said to be *ergodic* if, on average, it spends time in regions proportional to the utility of exploring said area
- For $t_f \to \infty$, ergodic trajectories guarantee complete coverage

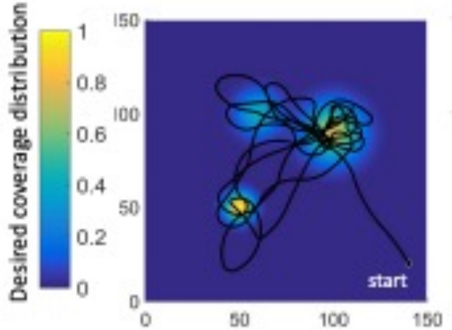
$$\lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} \phi(g \circ x(t))dt = \int_{\mathcal{W}} \phi(w)\mu(w)dw$$

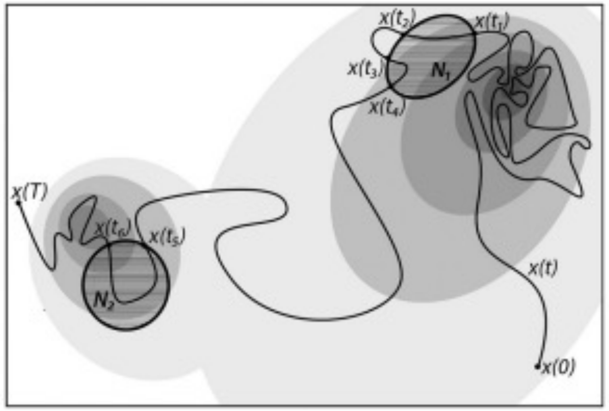




[1] Ayvali, Elif, Hadi Salman, and Howie Choset. "Ergodic coverage in constrained environments using stochastic trajectory optimization." 2017 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS). IEEE, 2017.
[2] Miller, Lauren M., et al. "Ergodic exploration of distributed information." IEEE Transactions on Robotics 32.1 (2015): 36-52.

Introduction to Ergodic Coverage





Ergodic Coverage

- An ergodic metric computes the distance to ergodicity
- Can be computed in the Fourier domain
  - Trajectory measure defined as Dirac delta function

$$\mathcal{E}_\mu(x) = \sum_{k \in \mathcal{K}^v} \Lambda_k \left(c_{t_f}^k - \mu^k \right)^2$$

$$= \sum_{k \in \mathcal{K}^v} \Lambda_k \left(\frac{1}{t_f} \int_0^{t_f} F_k(g \circ x(t))dt - \int_{\mathcal{W}} F_k(w)\mu(w)dw \right)^2$$

[1] Ayvali, Elif, Hadi Salman, and Howie Choset. "Ergodic coverage in constrained environments using stochastic trajectory optimization." 2017 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS). IEEE, 2017.
[2] Miller, Lauren M., et al. "Ergodic exploration of distributed information." IEEE Transactions on Robotics 32.1 (2015): 36-52.

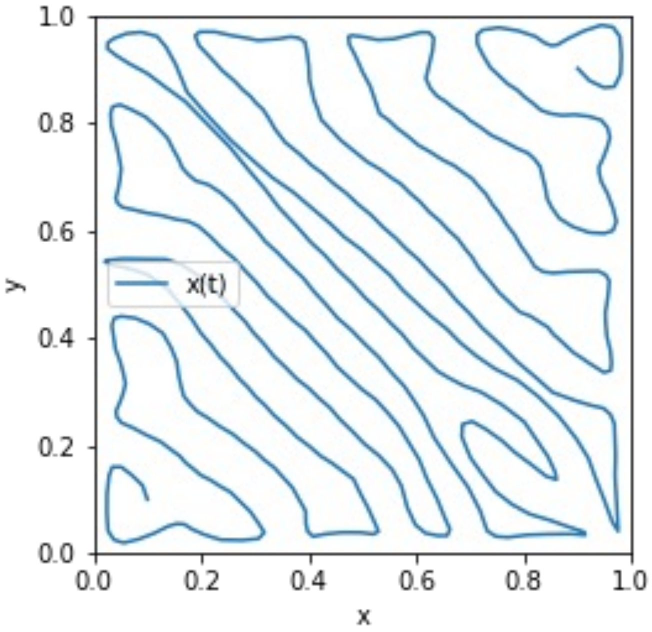
Ergodic Trajectory Optimization

Ergodic Trajectory Optimization

- Ergodic metric can be included in common trajectory optimization formulations

$$\min_{x(t),u(t)} \left\{ \mathcal{E}(x(t), \mu) + \int_0^{t_f} u(t)^\top \mathbf{R} u(t) dt \right\}$$

$$\text{s.t.} \quad \begin{cases} x(t) \in \mathcal{X}, u(t) \in \mathcal{U}, g \circ x(t) \in \mathcal{W}, \ \forall t \\ x(0) = x_0, x(t_f) = x_f \\ \dot{x} = f(x, u) \\ h_1(x, u) \leq 0, h_2(x, u) = 0 \end{cases}$$



Uniform coverage on 2D workspace based on continuous ergodic coverage

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Approach 1: iLQR, DDP

$$\mathcal{E}_\mu(x) = \sum_{k\in\mathcal{K}^v} \Lambda_k \left(c^k_{t_f} - \mu^k\right)^2$$

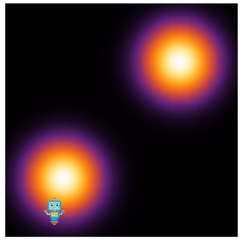
$$= \sum_{k\in\mathcal{K}^v} \Lambda_k \left(\frac{1}{t_f}\int_0^{t_f} F_k(g\circ x(t))dt - \int_{\mathcal{W}} F_k(w)\mu(w)dw\right)^2$$

Requires First Derivative Information

- Problem not in Bolza form
- Its derivative is!

$$D\mathcal{E}_\mu(x(t))\cdot\delta x(t) = 2\sum_{k\in\mathcal{K}^v} \Lambda_k \left(c^k_{t_f} - \mu^k\right)\cdot\frac{1}{t_f}\int_0^{t_f} DF_k(g\circ x(t))\cdot\delta x(t)\,dt$$

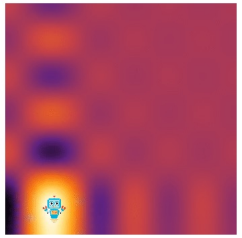
$$\implies 2\sum_{k\in\mathcal{K}^v} \Lambda_k \left(c^k_{t_f} - \mu^k\right)\cdot\frac{1}{t_f}DF_k(g\circ x(t))\cdot\delta x$$



$$DF_k(g\circ x) = \frac{dF_k(w)}{dw}\frac{dg(x)}{dx}$$

Plug and Play Into Traj Opt. Solver

- Add in control costs as needed
- Choose your integration



[1] Miller, Lauren M., et al. "Ergodic exploration of distributed information." IEEE Transactions on Robotics 32.1 (2015): 36-52.

Approach 2: Discretize then Optimize

Approach

- We solve problem over discrete trajectories
- Form a discrete-time ergodic metric

$$\mathbf{x} = [x_0, x_1, \dots, x_{T-1}]$$

$$\mathbf{u} = [u_0, u_1, \dots u_{T-1}]$$

$$T = \frac{t_f}{dt}$$

$$x_{t+1} = F(x_t, u_t) = x_t + \Delta t f(x_t, u_t)$$

$$\mathcal{E}(x(t), \mu) \approx \hat{\mathcal{E}}(\mathbf{x}, \mu) = \sum_{k \in \mathcal{K}^v} \Lambda_k \left(c^k(\mathbf{x}) - \mu^k\right)^2$$

$$= \sum_{k \in \mathcal{K}^v} \Lambda_k \left(\frac{1}{T} \sum_{t=0}^{T-1} \underline{F_k(g \circ x_t))} - \mu^k\right)^2$$

[1] Lerch, Cameron, Dayi Dong, and Ian Abraham. "Safety-critical ergodic exploration in cluttered environments via control barrier functions." In 2023 IEEE International Conference on Robotics and Automation (ICRA), pp. 10205-10211. IEEE, 2023.

Approach 2: Discretize then Optimize

Approach

- We solve problem over discrete trajectories
- Form a discrete-time ergodic NLP

$$\min_{\mathbf{x},\mathbf{u}} \quad \hat{\mathcal{E}}(\mathbf{x},\mu) + \sum_{t=0}^{T-1} \ell(x_t, u_t)$$

$$\text{s.t.} \quad \begin{cases} x_t \in \mathcal{X}, u_t \in \mathcal{U}, g(x_t) \in \mathcal{W}, \forall t \\ x_0 = \tilde{x}_0, x_T = \tilde{x}_f \\ x_{t+1} = x_t + \Delta t f(x_t, u_t) \\ h_1(\mathbf{x},\mathbf{u}) \leq 0, h_2(\mathbf{x},\mathbf{u}) = 0 \end{cases}$$

$$\mathcal{E}(x(t),\mu) \approx \hat{\mathcal{E}}(\mathbf{x},\mu) = \sum_{k\in\mathcal{K}^v} \Lambda_k \left(c^k(\mathbf{x}) - \mu^k\right)^2$$

$$= \sum_{k\in\mathcal{K}^v} \Lambda_k \left(\frac{1}{T}\sum_{t=0}^{T-1} F_k(g\circ x_t)) - \mu^k\right)^2$$

Plug and Play Into Opt. Solver

- Hessian not very friendly
- Approx. Newton’s method works
- LBFGS
- Choose your own discretization/colocation

[1] Lerch, Cameron, Dayi Dong, and Ian Abraham. "Safety-critical ergodic exploration in cluttered environments via control barrier functions." In 2023 IEEE International Conference on Robotics and Automation (ICRA), pp. 10205-10211. IEEE, 2023.

Approach 3: Sample-based (TBD Later)

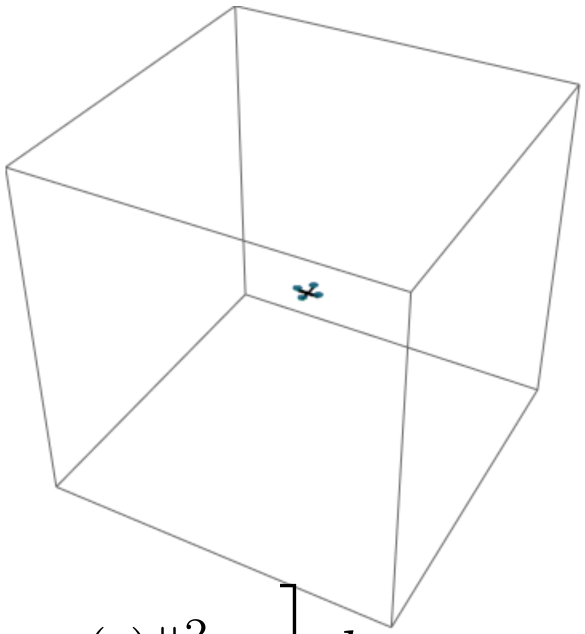
Approach

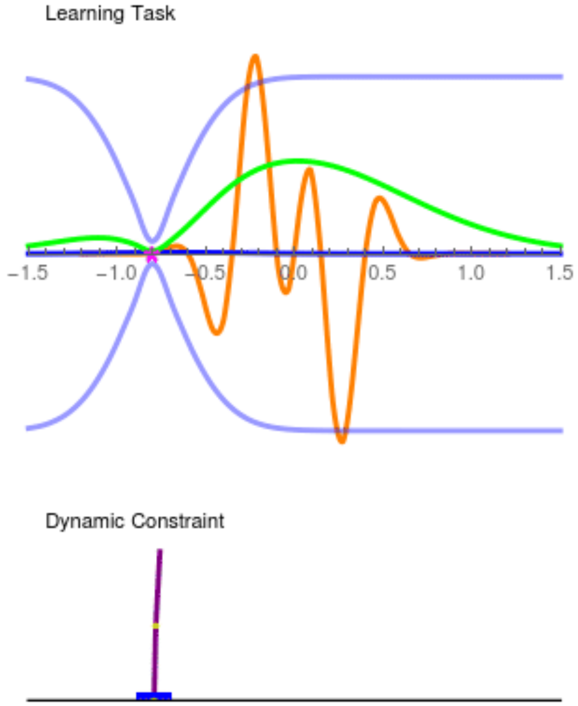
- Optimize problem through sample-based optimization!
- Very fast, but less accurate

$$D_{KL}(p\|q) = -E_{p(s)}[\log q(s)]$$

$$\approx -\sum_{i=1}^{N} p(s_i) \log q(s_i)$$

$$\propto -\sum_{i=1}^{N} p(s_i) \log \int_{t_0}^{t_f} \exp\left[-\frac{1}{2}\|s_i - \bar{x}(t)\|^2_{\Sigma^{-1}}\right] dt,$$





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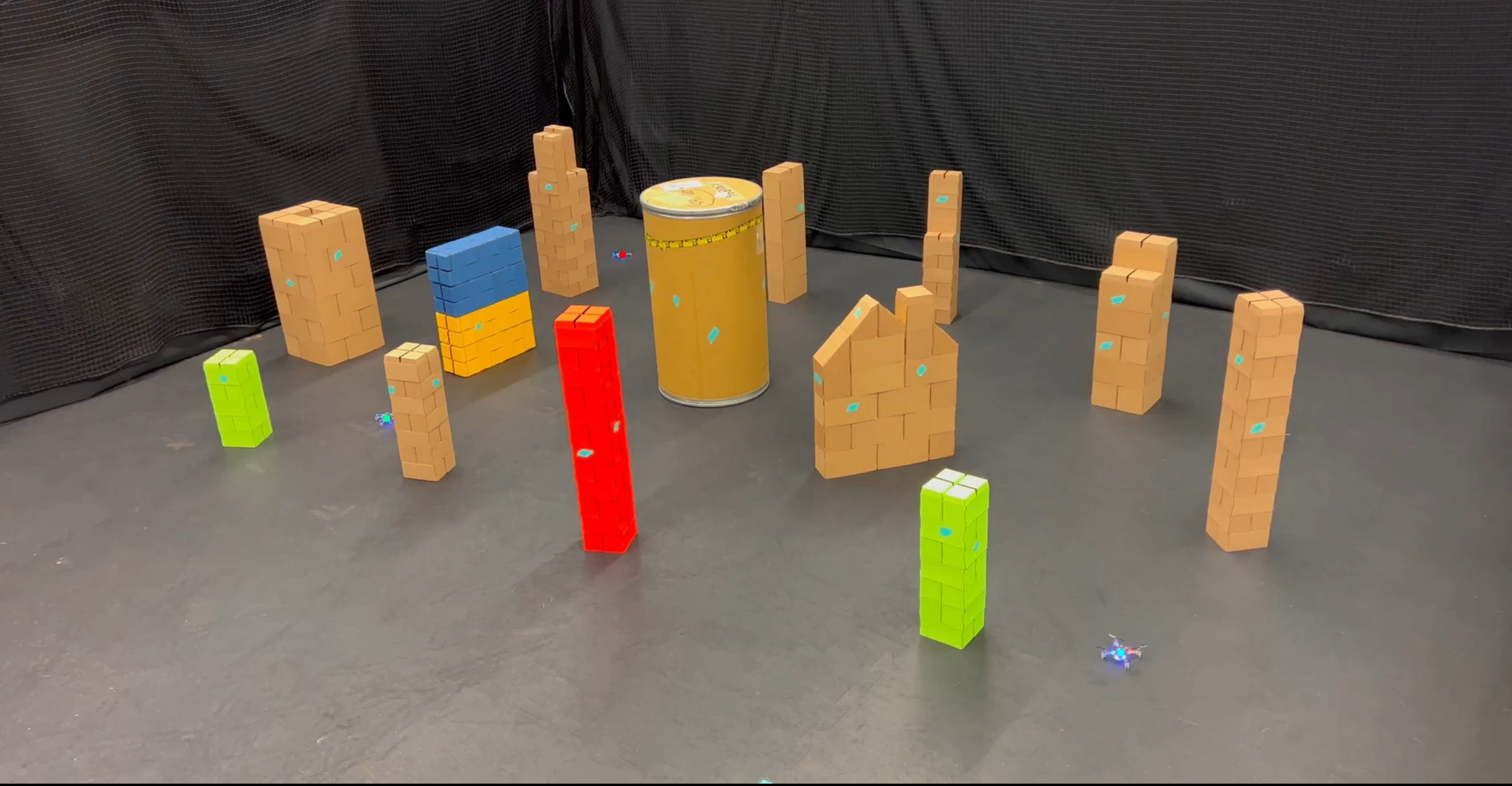
Uniform Coverage with Multiple Drones and Safety Constraints

Multi-drone ergodic search with drone-drone and drone-obstacle avoidance using



Multi-drone ergodic search with drone-drone and drone-obstacle avoidance using CBF