Ergodic control using diffusion, HEDAC

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[MM09] and [MM11] define the coverage as a distribution in the domain $\Omega \subset \mathbb{R}^2$ using convolution of $\delta(x)$ Dirac delta function:

$$c_{smc}(\mathbf{x}) = \frac{1}{N\,t} \sum_{i=1}^{N} \int_0^t \delta(\mathbf{x} - \mathbf{z}_i(\tau))\, \mathrm{d}\tau,$$

SMC motion control:

$$\frac{\mathrm{d}\mathbf{z}_i}{\mathrm{d}t} = \sum\nolimits_{\mathbf{k}} \Lambda(\mathbf{k}) \; (m_{\mathbf{k}} - c_{\mathbf{k}}) \; \nabla f_{\mathbf{k}}(\mathbf{z}_i(t))$$

Velocity (Change of direction)	Weights	The difference of Fourier coeffitients	Gradient of Fourier basis function at current location

A spectral-based smoothing:

$$\Lambda(\mathbf{k}) = \frac{1}{\left(1 + \|\mathbf{k}\|^2\right)^{\frac{n+1}{2}}}$$

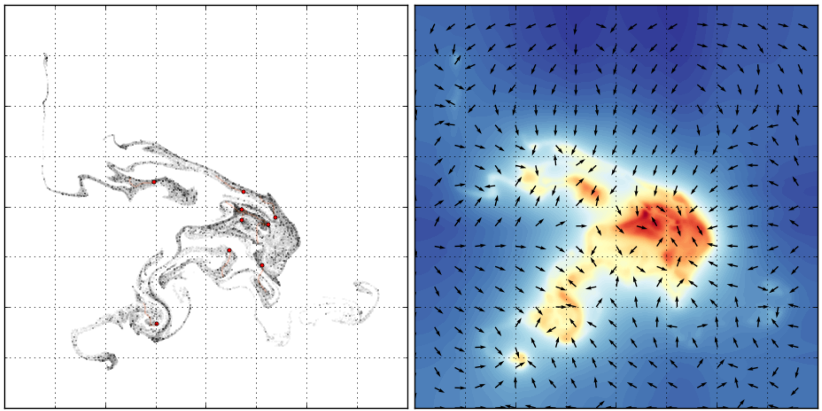
[MM09] George Mathew and Igor Mezić. “Spectral multiscale coverage: A uniform coverage algorithm for mobile sensor networks”. In: *Proceedings of the 48th IEEE Conference on Decision and Control* (2009), pp. 7872–7877

[MM11] George Mathew and Igor Mezić. “Metrics for ergodicity and design of ergodic dynamics for multi-agent systems”. In: *Physica D: Nonlinear Phenomena* 240.4-5 (2011), pp. 432–442

One can easily construct an attracting field (not only gradients at agents' current locations):

$$s(\mathbf{x}) = m(\mathbf{x}) - c(\mathbf{x})$$

$$u(\mathbf{x}) = \sum_{\mathbf{k}} \Lambda(\mathbf{k})\, s_{\mathbf{k}}\, f_{\mathbf{k}}(\mathbf{x}),$$



SMC has some nice feautes/extensions:

- Good ballance between local and global coverage
- Solid and robust convergence
- Applicable for different motion models (constraints?)
- It can be applied to dynamically alternated goal distribution $m(\mathbf{x}, t)$ [MSM10]
- MH370 search application, [Ivi+20]

But there are some limits:

- Does not acknowledge the "reach" of the kernel function γ
- Accomplished paths create artificial barriers (broad exploration due to $m - c < 0$)
- Limited to rectangular domains
- Difficult to consider obstacles

[MSM10] George Mathew, Amit Surana, and Igor Mezić. “Uniform coverage control of mobile sensor networks for dynamic target detection”. In: IEEE. 2010, pp. 7292–7299

[Ivi+20] S. Ivić et al. “Search strategy in a complex and dynamic environment: the MH370 case”. In: *Scientific reports* 10.1 (2020), pp. 1–15

Introducing radial basis kernel function

If sensing function γ is a positive, smooth and scalable (σ) local Radial Basis Function that satisfy:

$$\int_{\mathbb{R}^n} \gamma_\sigma(\mathbf{r})\,\mathrm{d}\mathbf{r} = 1,$$

with known agent trajectories, the coverage density at time t can be defined as:

$$\tilde{c}_\sigma(\mathbf{x}, t) = \frac{1}{N\,t}\sum_{i=1}^{N}\int_0^t \gamma_\sigma(\mathbf{x} - \mathbf{z}_i(\tau))\,\mathrm{d}\tau.$$

When observing the γ in the limit $\sigma \to 0$:

$$\lim_{\sigma\to 0} \gamma_\sigma(\mathbf{r}) = \lim_{\sigma\to 0} \sigma^{-n}\gamma\left(\frac{\mathbf{r}}{\sigma}\right) = \delta(\mathbf{r}).$$

the coverage function will became a distribution [ICM16]:

$$\lim_{\sigma\to 0} c_\sigma(\mathbf{x}) = c_{smc}(\mathbf{x})$$

[ICM16] S. Ivić, B. Crnković, and I. Mezić. “Ergodicity-based cooperative multiagent area coverage via a potential field”. In: *IEEE transactions on cybernetics* 47.8 (2016), pp. 1983–1993

Introducing radial basis kernel function

Easy to incorporate in SMC algorithm: c_k are obtained with FFT on the field c_σ

```
xmg, ymg = numpy.meshgrid(x, y)
c = np.zeros([ny, nx])
for p in points:
    c += numpy.exp(- ((xmg - p[0]) ** 2 + (ymg - p[1]) ** 2) / (2 * sigma ** 2))
```

Spectral smoothing and attracion field

```
scipy.fftpack import dct, idct
s = c - m
s[s < 0] = 0 # Comment line for original SMC formulation
s_k = dct(dct(s.T, norm='ortho').T, norm='ortho')

# Smoothing
ky, kx = numpy.meshgrid(numpy.arange(ny), numpy.arange(nx))
u_k = 1.0 / (1 + kx ** 2 + ky ** 2) ** 1.5 * s_k.T
# Obtain u
u = idct(idct(u_k.T, norm='ortho').T, norm='ortho')

vy, vx = numpy.gradient(self.U)
```

Heat Equation Driven Area Coverage (HEDAC)

- Adopts the attraction field approach from SMC [ICM16]
- Modeling the potential field via the stationary 2D PDE heat equation (Helmholtz equation)

$$\underbrace{\alpha \cdot \frac{\partial^2 u}{\partial \mathbf{x}^2}}_{\text{Conduction (diffusion)}} + \underbrace{s}_{\text{heat (source)}} \underbrace{- \beta \cdot u}_{\text{cooling (sink)}} = 0$$

with Neumann boundary conditions:

$$\underbrace{\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\Gamma} = 0}_{\text{insulation}}$$

- The difference between achieved and goal density $s = \max(m - c, 0)$ acts as a heat/potential source
- Resulting potential u is "similar" to the source s (smoothing, gradient)
- Physical interpretation and intuition
- Why stationary?

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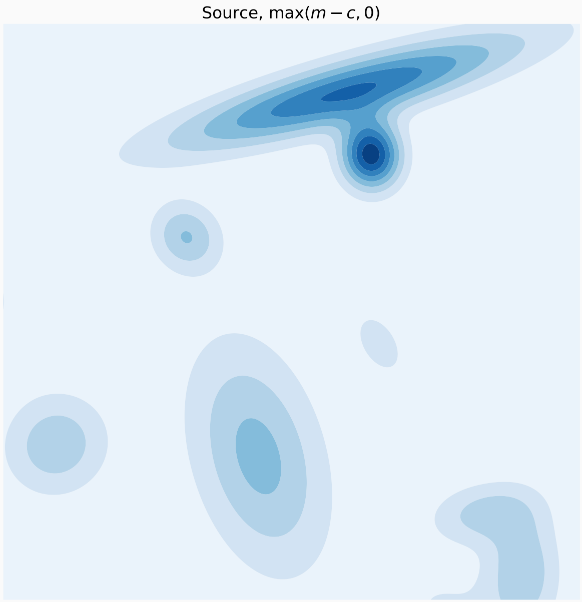
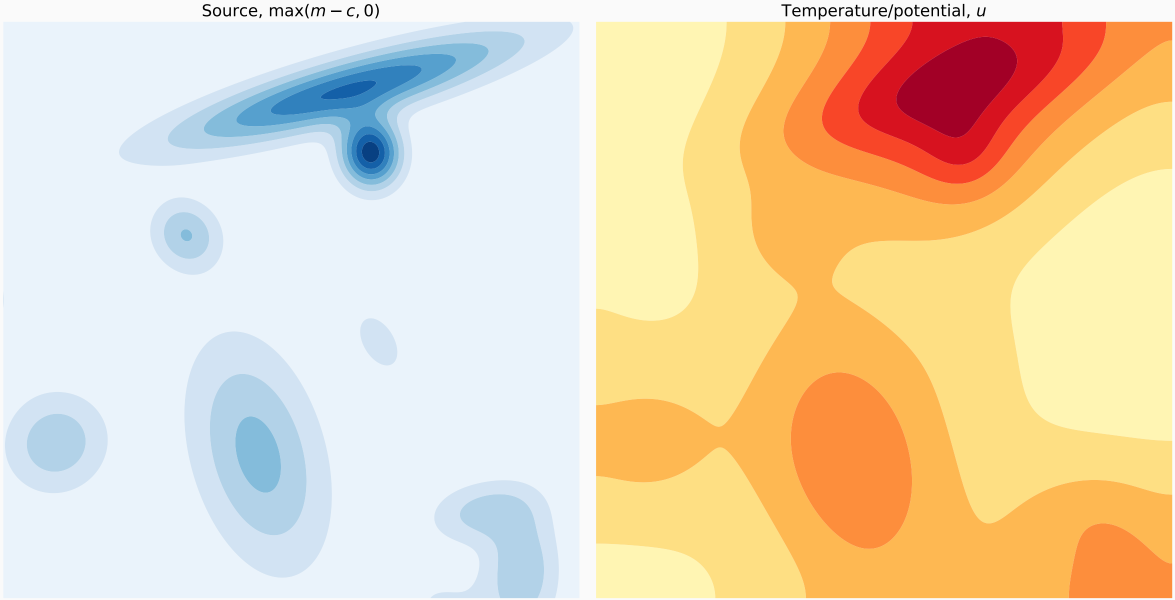
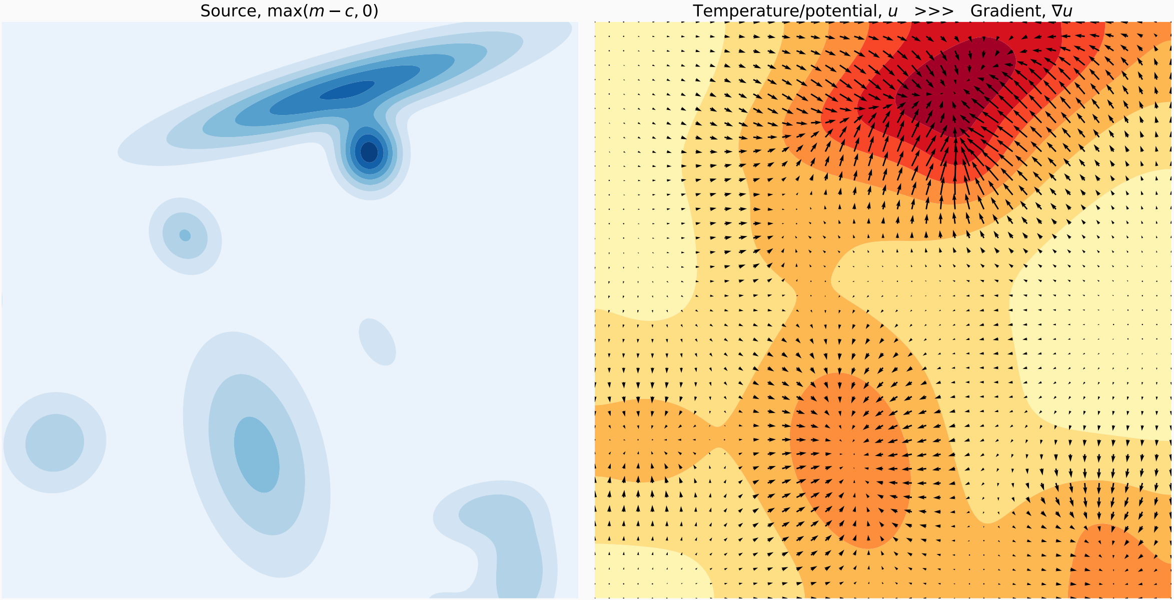

Source, max($m-c$, 0)



Source, max$(m-c, 0)$
Temperature/potential, u


Source, max($m-c$, 0)
Temperature/potential, u >>> Gradient, ∇u

Motion control is established via a feedback loop:

$$\alpha \cdot \frac{\partial^2 u}{\partial \mathbf{x}^2} + s - \beta \cdot u = 0,$$

$$c(\mathbf{x}, t) = \frac{1}{N\,t} \sum_{i=1}^{N} \int_0^t \gamma_\sigma(\mathbf{x} - \mathbf{z}_i(\tau))\, \mathrm{d}\tau,$$

$$\frac{\mathrm{d}\mathbf{z}_i}{\mathrm{d}t} = v_a \cdot \frac{\nabla u(\mathbf{z}_i(t))}{|\nabla u(\mathbf{z}_i(t))|}$$

- General coverage [ICM16]:
    - Uniform
    - Nonuniform
- *Drawing without lifting the pen from the paper* (unpublished)

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FDM scheme

PDE:

$$\alpha \cdot \frac{\partial^2 u}{\partial \mathbf{x}^2} \quad + \quad m \quad -\beta \cdot u \quad = \quad 0$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\alpha \cdot \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} \right) \quad + \quad m(x_{i,j}) \quad -\beta \cdot u_{i,j} \quad = \quad 0$$

for $i = 1, \ldots, n_x - 1, j = 1, \ldots, n_y - 1.$

BC:

$$\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\Gamma} = 0$$

$$\Downarrow$$

$$\frac{u_{1,j} - u_{0,j}}{\Delta x} = 0, \qquad \frac{u_{n_x,j} - u_{n_x-1,j}}{\Delta x} = 0, \qquad \text{for } j = 1, \ldots, n_y - 1$$

$$\frac{u_{i,1} - u_{i,0}}{\Delta x} = 0, \qquad \frac{u_{i,n_y} - u_{i,n_y-1}}{\Delta x} = 0 \qquad \text{for } i = 1, \ldots, n_x - 1.$$

Solution **u** is obtained by solving linear system

$$\mathbf{A} \cdot \mathbf{u} = \mathbf{b}(m)$$

Matrix **A** is constant:

$$\mathbf{u} = \mathbf{A}^{-1} \cdot \mathbf{b}(m)$$

which allows computationally efficient implementation.

```
# Inializtion
A = numpy.zeros((nx * ny, nx * ny))
b = numpy.zeros(nx * ny)
# ... populate system matrix A
invA = numpy.linalg.inv(A)

# Time loop
# ... update vector b (according to current coverage)
u = numpy.dot(A, b) # Solve potential u
```

FDM scheme

```
# Inializtion
A = scipy.sprse.lil_matrix((nx * ny, nx * ny))
b = np.zeros(nx * ny)
# ... populate system matrix A
A = A.tocsc()
lu = scipy.sparse.linalg.splu(A)

# Time loop
# ... update vector b (according to current coverage)
u = lu.solve(b) # Solve potential
```

References

- George Mathew and Igor Mezić. “Spectral multiscale coverage: A uniform coverage algorithm for mobile sensor networks”. In: *Proceedings of the 48th IEEE Conference on Decision and Control* (2009), pp. 7872–7877.
- George Mathew, Amit Surana, and Igor Mezić. “Uniform coverage control of mobile sensor networks for dynamic target detection”. In: IEEE. 2010, pp. 7292–7299.
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- S. Ivić, B. Crnković, and I. Mezić. “Ergodicity-based cooperative multiagent area coverage via a potential field”. In: *IEEE transactions on cybernetics* 47.8 (2016), pp. 1983–1993.
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