Ergodic control (SMC) in 1D

Notation and reference

RCFS

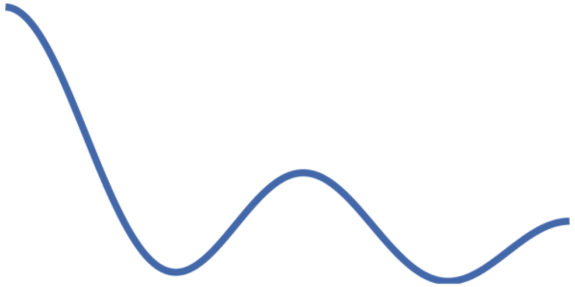
Robotics
Codes
From
Scratch.

https://rcfs.ch

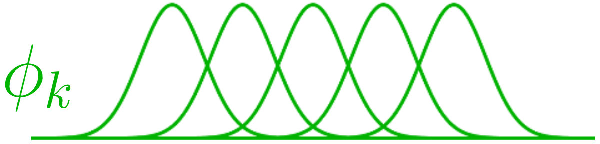
[Calinon, S. (2019). Mixture Models for the Analysis, Edition, and Synthesis of Continuous Time Series. Mixture Models and Applications, Springer]

Superposition with basis functions

$$g(t) = \sum_{k=1}^{K} w_k \, \phi_k(t) = \boldsymbol{w}^\top \boldsymbol{\phi}(t)$$

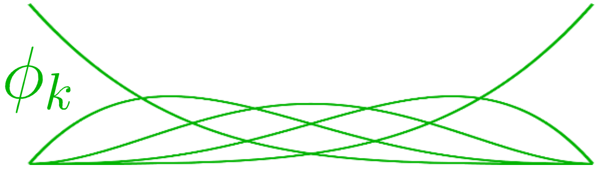


ϕ_k



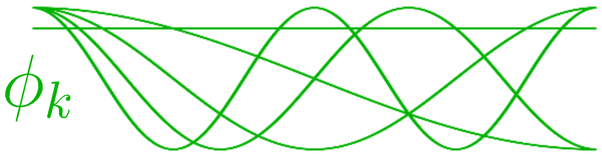
Radial basis functions

ϕ_k

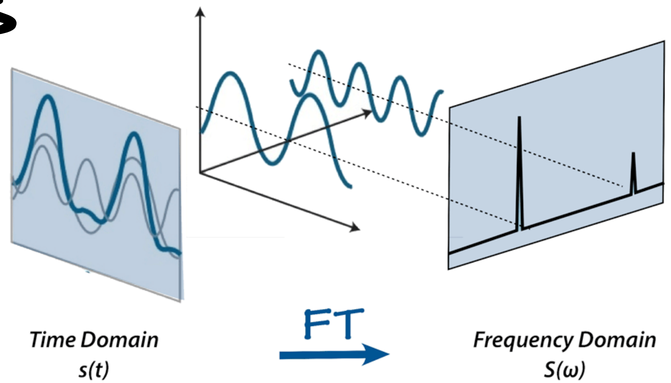


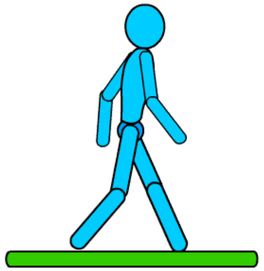
Bernstein polynomials

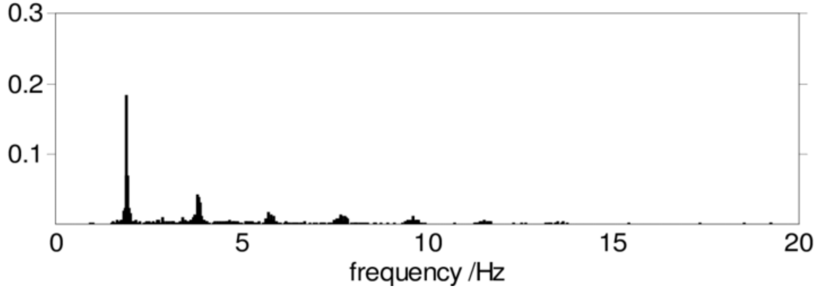
ϕ_k

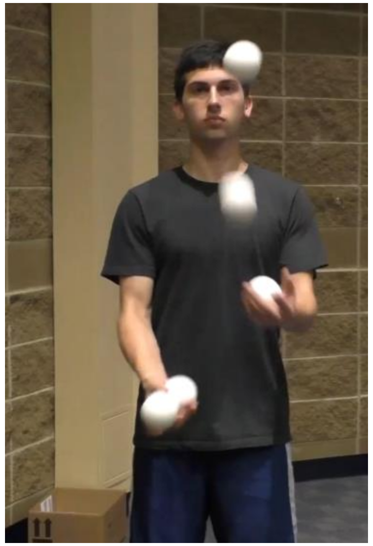


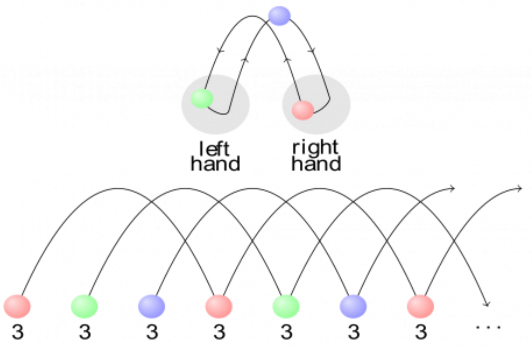
Fourier series



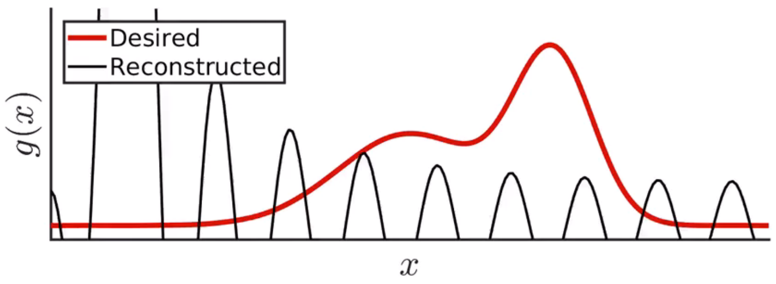


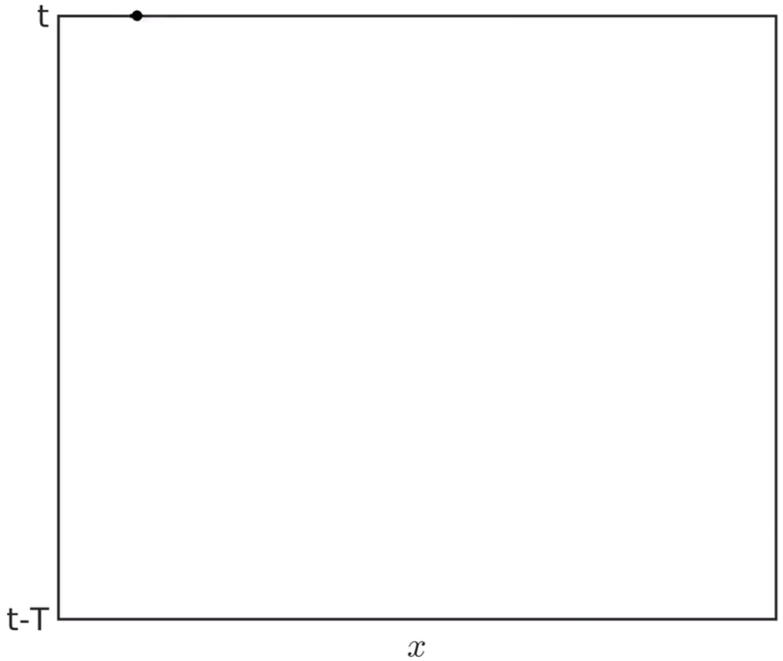


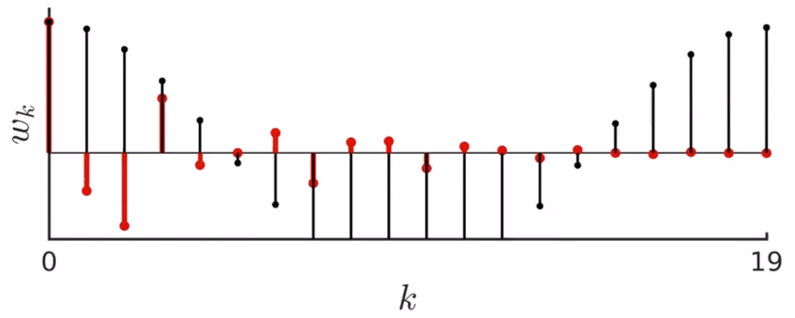


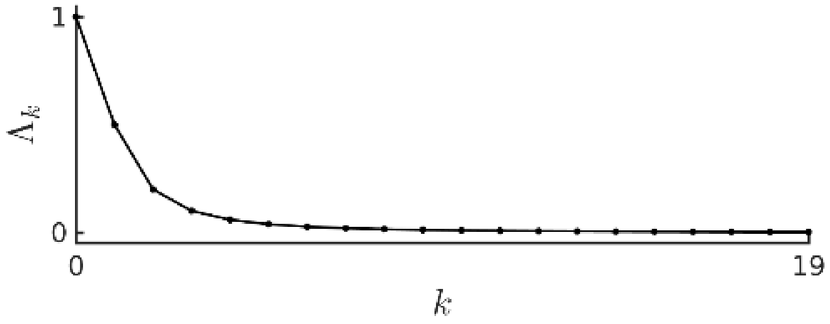


Ergodic control in 1 D









$$\min_{u(t)} \sum_{\boldsymbol{k}\in\mathcal{K}} \Lambda_{\boldsymbol{k}} \Big(w_{\boldsymbol{k}} - \hat{w}_{\boldsymbol{k}}\Big)^2$$

$$u(t) = \tilde{u}(t)\frac{u^{\max}}{\|\tilde{u}(t)\|}, \quad \text{with} \quad \tilde{u} = -\sum_{\boldsymbol{k}\in\mathcal{K}} \Lambda_{\boldsymbol{k}} \left(w_{\boldsymbol{k}} - \hat{w}_{\boldsymbol{k}}\right) \frac{\partial}{\partial x}\phi_{\boldsymbol{k}}\big(x(t)\big)$$

Fourier series: Symmetry property

$$g(x) = \sum_k w_k \, \phi_k(x) = \boldsymbol{w}^\top \boldsymbol{\phi}(x)$$

$$\phi_k(x) = \frac{1}{L} \exp\left(-i \frac{2\pi k x}{L}\right) = \frac{1}{L}\left(\cos\left(\frac{2\pi k x}{L}\right) - i \, \sin\left(\frac{2\pi k x}{L}\right)\right)$$

Symmetry property:

If $g(x)$ is real and even, $\phi_k(x)$ is also real and even.

It then simplifies to $\phi_k(x) = \frac{1}{L} \cos\left(\frac{2\pi k x}{L}\right)$.

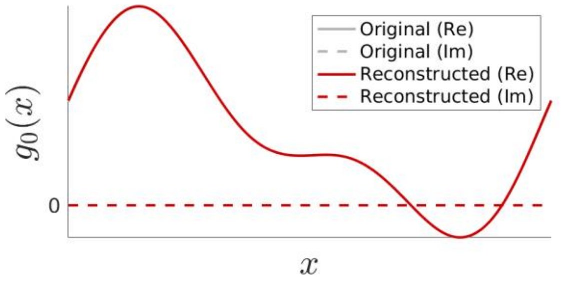
In practice, we only need to evaluate on the range $k \in [0, \ldots, K-1]$, as the basis functions are even.

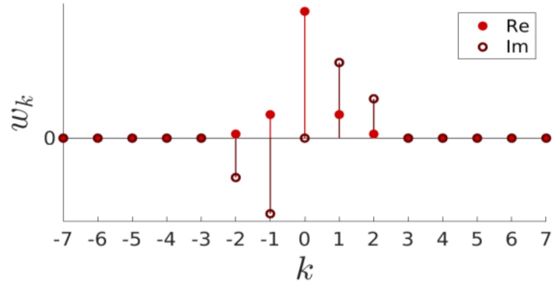
We then have $g(x) = w_0 + \sum_{k=1}^{K-1} w_k \, 2 \cos\left(\frac{2\pi k x}{L}\right)$, by exploiting $\cos(0) = 1$.

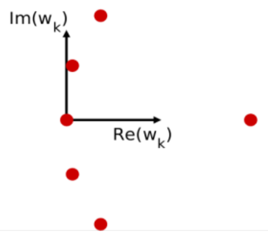
Fourier series: Symmetry property

$$g(x) = \sum_k w_k \, \phi_k(x) = \boldsymbol{w}^\top \boldsymbol{\phi}(x)$$

$$\phi_k(x) = \frac{1}{L} \exp\left(-i \frac{2\pi k x}{L}\right) = \frac{1}{L} \left(\cos\left(\frac{2\pi k x}{L}\right) - i \, \sin\left(\frac{2\pi k x}{L}\right) \right)$$







Fourier series: Gaussian property

$$g(x) = \sum_k w_k \, \phi_k(x) = \boldsymbol{w}^\top \boldsymbol{\phi}(x)$$

$$\phi_k(x) = \frac{1}{L} \exp\left(-i \frac{2\pi k x}{L}\right) = \frac{1}{L}\left(\cos\left(\frac{2\pi k x}{L}\right) - i \sin\left(\frac{2\pi k x}{L}\right)\right)$$

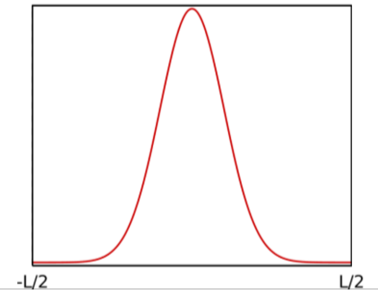
Gaussian property:

If $g_0(x) = \mathcal{N}(x \,|\, 0, \sigma^2) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp(-\frac{x^2}{2\sigma^2})$ is mirrored to create a real and even periodic function $g(x)$ of period $L \gg \sigma$, the corresponding Fourier series coefficients are of the form $w_k = \exp(-\frac{2\pi^2 k^2 \sigma^2}{L^2})$.

Fourier series: Gaussian property

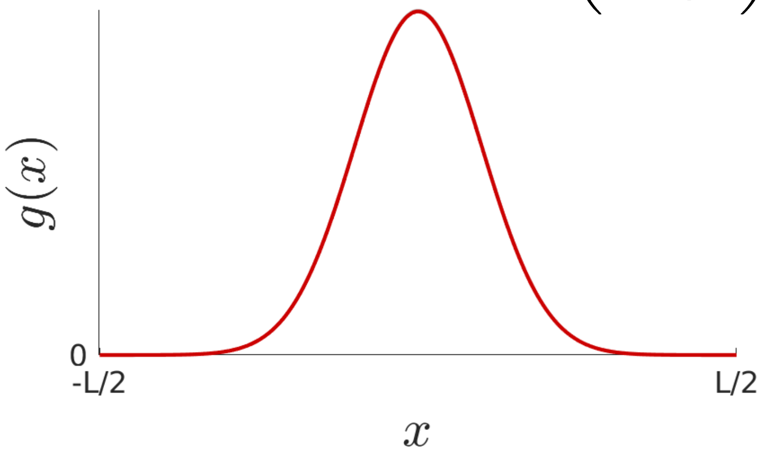
$$g(x) = \sum_k w_k\, \phi_k(x) = \boldsymbol{w}^\top \boldsymbol{\phi}(x)$$

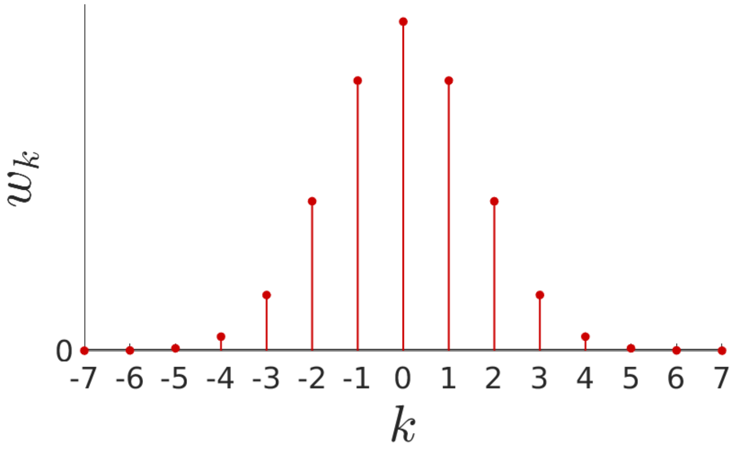
$$\phi_k(x) = \frac{1}{L}\exp\left(-i\frac{2\pi k x}{L}\right) = \frac{1}{L}\left(\cos\left(\frac{2\pi k x}{L}\right) - i\,\sin\left(\frac{2\pi k x}{L}\right)\right)$$



$$g(x) \sim \mathcal{N}(0, \sigma^2) \sim (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

$$w_k = \exp\left(-\frac{2\pi^2 k^2 \sigma^2}{L^2}\right)$$





Fourier series: Sum property

$$g(x) = \sum_k w_k \, \phi_k(x) = \boldsymbol{w}^\top \boldsymbol{\phi}(x)$$

$$\phi_k(x) = \frac{1}{L} \exp\left(-i \frac{2\pi k x}{L}\right) = \frac{1}{L}\left(\cos\left(\frac{2\pi k x}{L}\right) - i \, \sin\left(\frac{2\pi k x}{L}\right)\right)$$

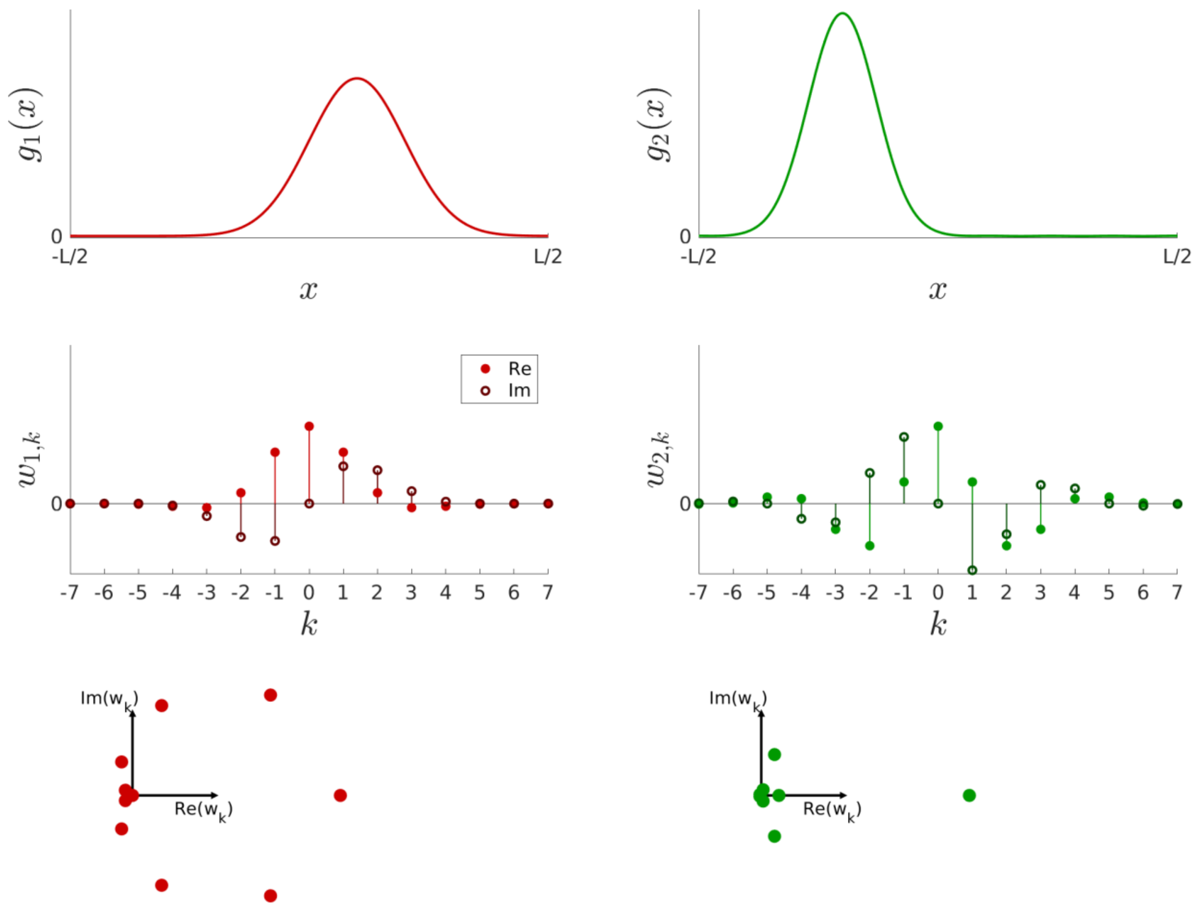
Combination property:

If $w_{k,1}$ (resp. $w_{k,2}$) are the Fourier series coefficients of a function $g_1(x)$ (resp. $g_2(x)$), then $\alpha_1 w_{k,1} + \alpha_2 w_{k,2}$ are the Fourier coefficients of $\alpha_1 g_1(x) + \alpha_2 g_2(x)$.

Fourier series: Sum property

$$g(x) = \sum_k w_k \, \phi_k(x) = \boldsymbol{w}^\top \boldsymbol{\phi}(x)$$

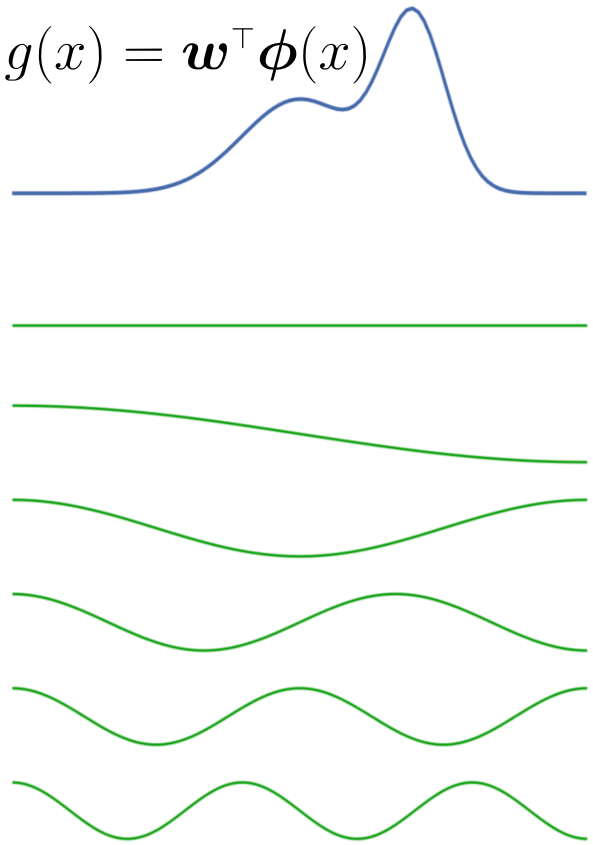
$$\phi_k(x) = \frac{1}{L} \exp\left(-i\frac{2\pi k x}{L}\right) = \frac{1}{L}\left(\cos\left(\frac{2\pi k x}{L}\right) - i \, \sin\left(\frac{2\pi k x}{L}\right)\right)$$



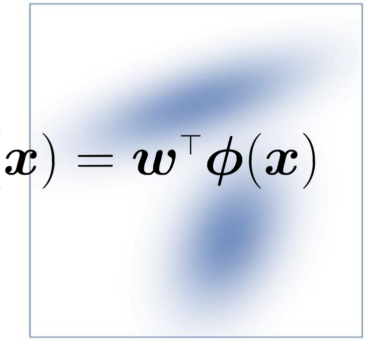
Ergodic control (SMC) in 2 D

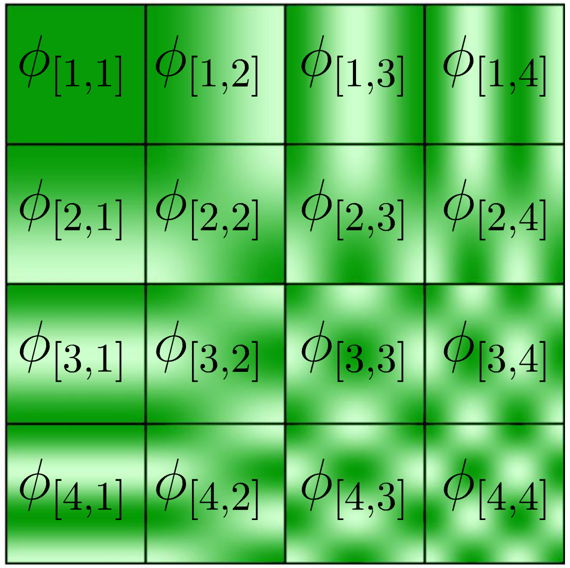
Ergodic control in 2 D

$$g(x) = \boldsymbol{w}^\top \boldsymbol{\phi}(x)$$



$$g(\boldsymbol{x}) = \boldsymbol{w}^\top \boldsymbol{\phi}(\boldsymbol{x})$$

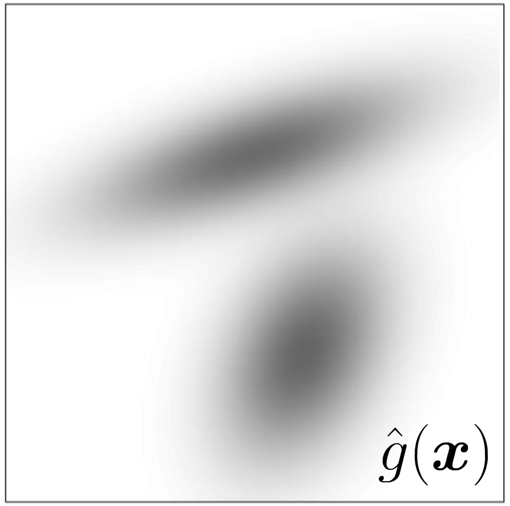




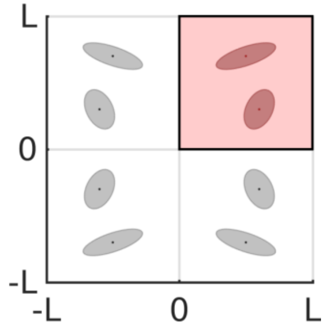
$$\phi_k(x) = \frac{1}{L} \exp\left(-i \frac{2\pi k x}{L}\right)$$

$$\phi_{\boldsymbol{k}}(\boldsymbol{x}) = \frac{1}{L^D} \prod_{d=1}^{D} \exp\left(-i \frac{2\pi k_d x_d}{L}\right)$$

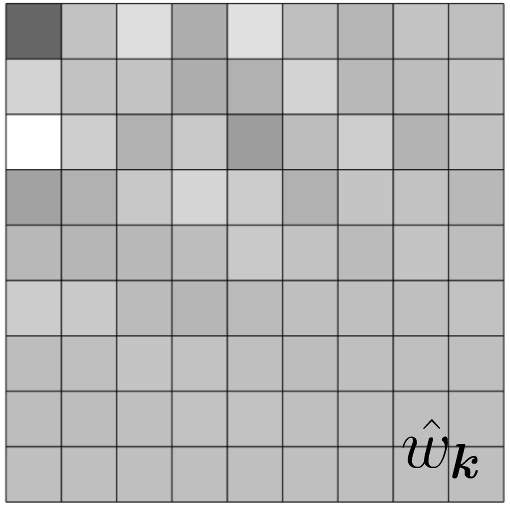
Ergodic control in 2 D



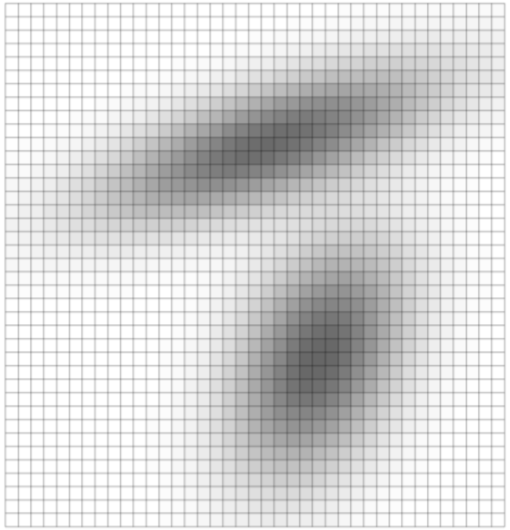
Gaussian property
Symmetry property
Shift property
Combination property



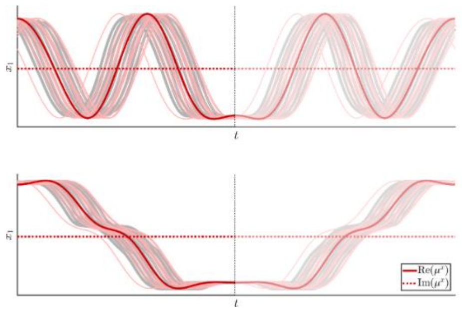
$\phi_{\boldsymbol{k}}(\boldsymbol{x})$ real and even



Discretization

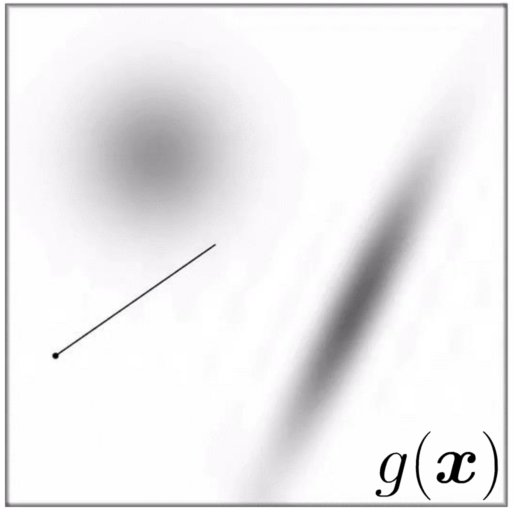


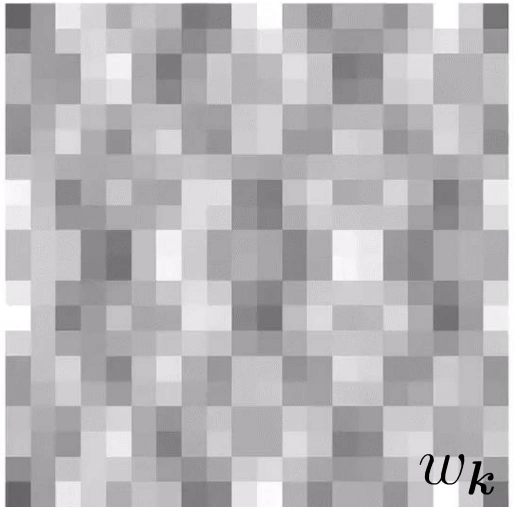
$$\hat{w}_{\boldsymbol{k}} = \int_{\boldsymbol{x}\in\mathcal{X}} \hat{g}(\boldsymbol{x})\ \phi_{\boldsymbol{k}}(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x}$$

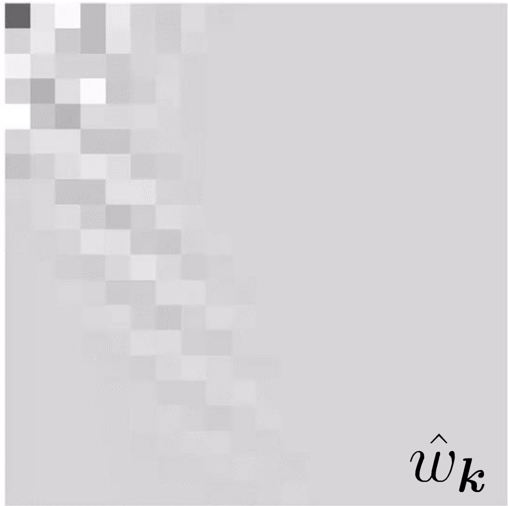


Generating a symmetric signal in 1D

Ergodic control in 2 D







$$\min_{\boldsymbol{u}(t)} \sum_{\boldsymbol{k}\in\mathcal{K}} \Lambda_{\boldsymbol{k}} \Big(w_{\boldsymbol{k}} - \hat{w}_{\boldsymbol{k}} \Big)^2$$

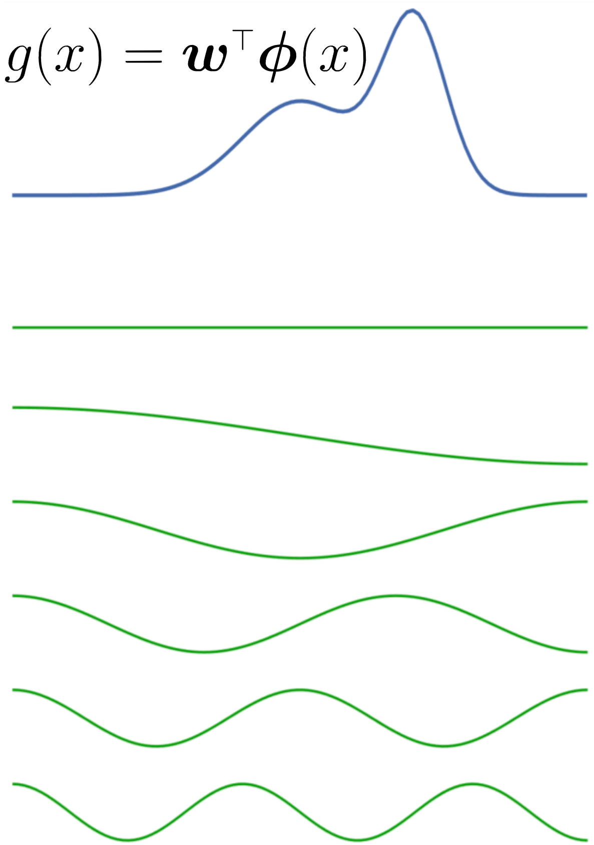
$$\boldsymbol{u} = \tilde{\boldsymbol{u}}(t) \frac{u^{\max}}{\|\tilde{\boldsymbol{u}}(t)\|}, \quad \text{with} \quad \tilde{\boldsymbol{u}} = -\sum_{\boldsymbol{k}\in\mathcal{K}} \Lambda_{\boldsymbol{k}} \, \big(w_{\boldsymbol{k}} - \hat{w}_{\boldsymbol{k}}\big) \, \nabla_{\boldsymbol{x}} \phi_{\boldsymbol{k}}\big(\boldsymbol{x}(t)\big)^{\top}$$

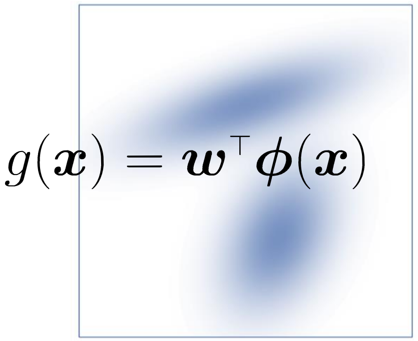
https://ergodiccontrol.github.io/sandbox.html

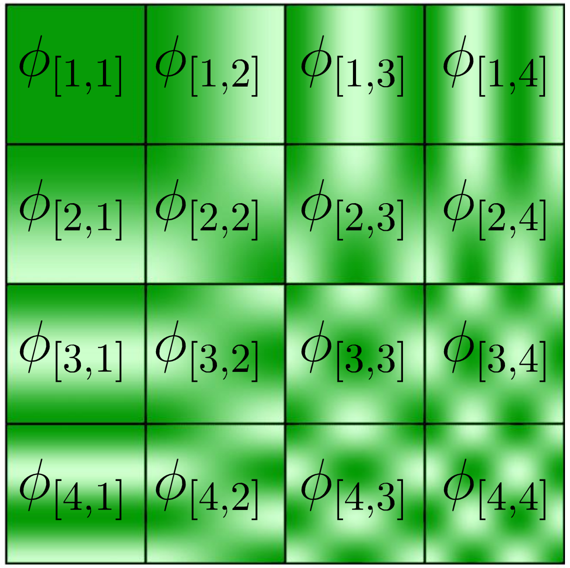
https://ergodiccontrol.github.io/sandbox3d.html

Ergodic control (SMC) in 6 D

Ergodic control for more than 3 dimensions







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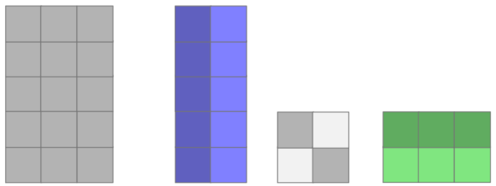
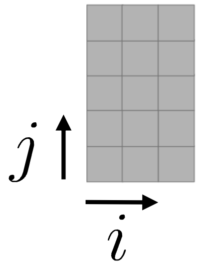
Separation of variables: a factorization problem

Matrix factorization with standard linear algebra:



$$\boldsymbol{X} = \boldsymbol{U} \; \boldsymbol{\Sigma} \; \boldsymbol{V}^\top$$

(singular value decomposition)



Rank-1 decomposition:

$$\boldsymbol{X}_{i,j} = \boldsymbol{U}_i \boldsymbol{V}_j$$

→ Representation in a separable form

Rank-R decomposition:

$$\boldsymbol{X}_{i,j} = \sum_{r=1}^{R} \boldsymbol{U}_{i,r} \boldsymbol{V}_{j,r}$$

$$\boldsymbol{X} = \boldsymbol{U}\boldsymbol{V}^\top$$

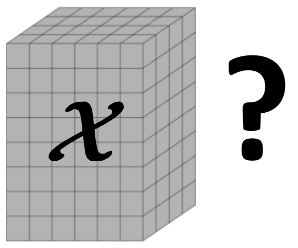
(in matrix form)

Extension to data with more indices (tensors):

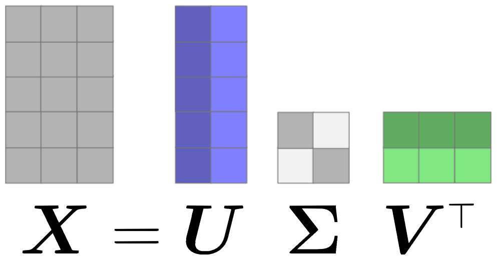
$$\boldsymbol{X}_{i,j,k,\ldots} = \sum_{r=1}^{R} \boldsymbol{U}_{i,r} \boldsymbol{V}_{j,r} \boldsymbol{W}_{k,r} \cdots$$

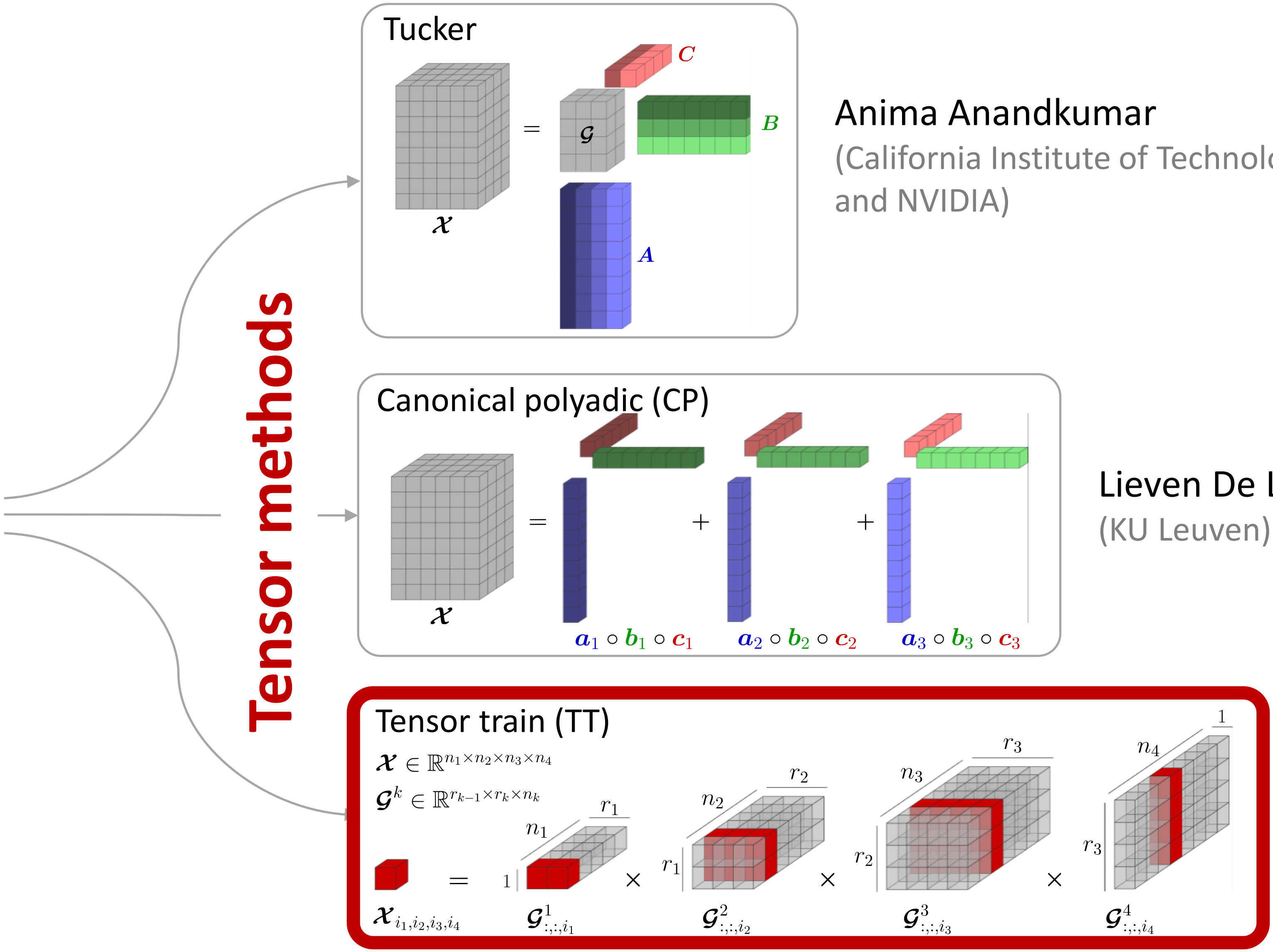
(CP decomposition)

Data structured as tensors



Matrix factorization with standard linear algebra:





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Ergodic exploration (Spectral Multiscale Coverage)

Control Command: $u_j(t) \propto \left\langle \boldsymbol{\Lambda} * (\hat{\mathcal{W}} - \mathcal{W}(t)), \nabla_j \boldsymbol{\Phi}\left(\boldsymbol{x}(t)\right) \right\rangle$

Ergodic control can scale!

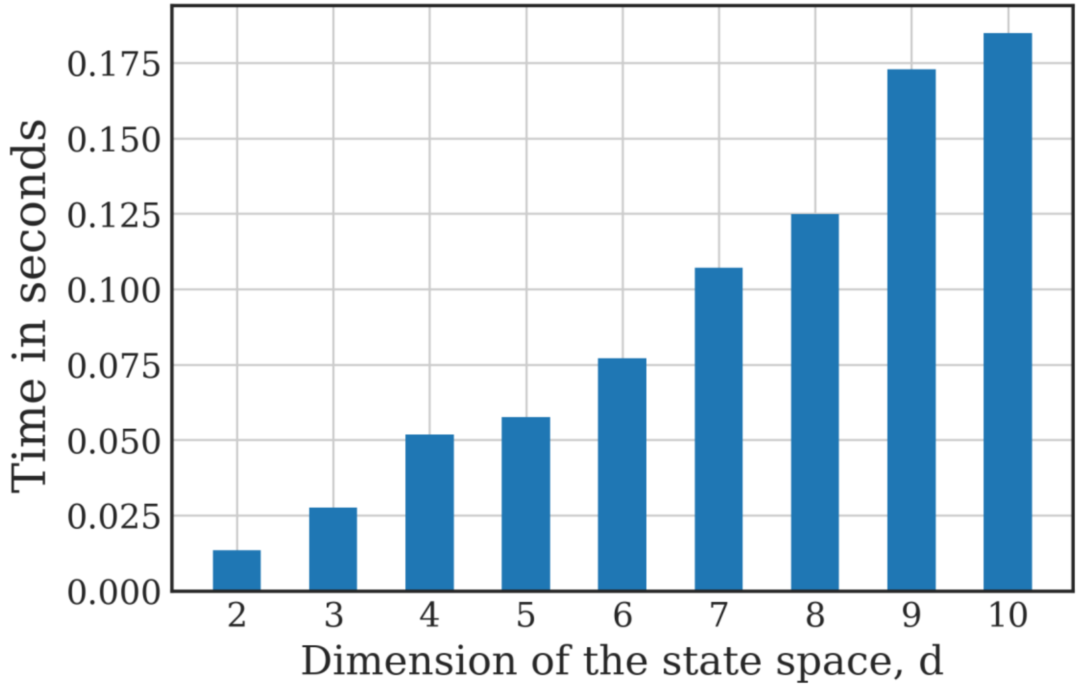
		Curse of Dimensionality Storage & Computation	Using Tensor Train (TT)
Weight Tensor:	$\boldsymbol{\Lambda}$	✘ $\mathcal{O}(K^d)$	✓ $\mathcal{O}(Kd)$
Basis Function Gradient:	$\nabla_j \boldsymbol{\Phi}\left(\boldsymbol{x}(t)\right)$	✘ $\mathcal{O}(K^d)$	✓ $\mathcal{O}(Kd)$
Fourier Coefficients:	$\hat{\mathcal{W}}$	✘ $\mathcal{O}(K^d)$	✓ $\mathcal{O}(Kdr^2)$
	$\mathcal{W}(t)$	✘ $\mathcal{O}(K^d)$	✓ $\mathcal{O}(Kdr^2)$

$$\hat{\mathcal{W}}_{\boldsymbol{k}} = \int_{x_1=0}^{L} \cdots \int_{x_d=0}^{L} P(\boldsymbol{x})\ \boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{x})\ dx_1 \ldots dx_d$$

$$\mathcal{W}_{\boldsymbol{k}}(t) = \frac{1}{t} \int_{\tau=0}^{t} \boldsymbol{\Phi}_{\boldsymbol{k}}(\boldsymbol{x}(\tau)) d\tau$$

Computation time (for K=5)

Time taken to compute Fourier series coefficients



Time taken to compute control commands