Robustness and Optimality in Ergodic Control

Ian Abraham

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Acknowledgements


Dayi Dong


Henry Berger


Elena Wittemyer


Adam Seewald


Cameron Lerch

[Berger In Prep] Reachability Analysis for Ergodic Coverage Problems
[Seewald ICRA 24'] Energy-Aware Ergodic Search: Continuous Exploration for Multi-Agent Systems with Battery Constraints
[Wittemyer IROS 23'] Bi-Level Image-Guided Ergodic Exploration with Application to Planetary Rovers
[Lerch ICRA 23'] Safety-critical ergodic exploration in cluttered environments via control barrier functions
[Dong RSS 23'] Time Optimal Ergodic Search
[Abraham TASE 21'] An Ergodic Measure for Active Learning from Equilibrium.
[Abraham WAFR 20'] Active Area Coverage from Equilibrium
[Abraham RSS 18'] Data-Driven Measurement Models for Active Localization in Sparse Environments
[Abraham RAL 17'] Ergodic Exploration using Binary Sensing for Nonparametric Shape Estimation

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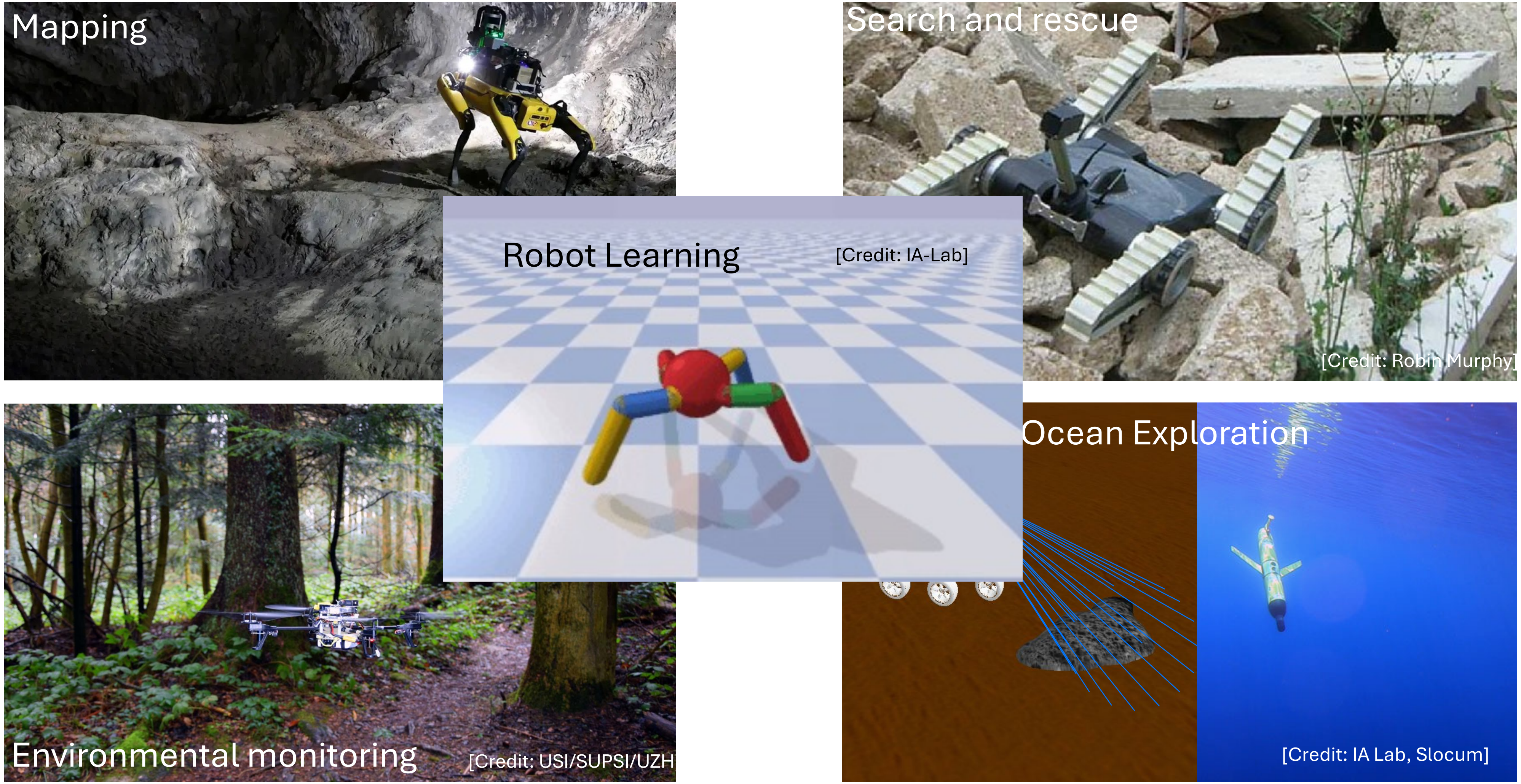
Motivation: Optimal and Robust Exploration



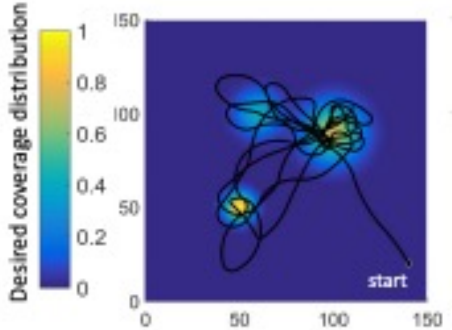
- Safety Guarantees in Ergodic Control
    - Set-Invariance with Control Barrier Functions
    - Hamilton Jacobi Isaacs Reachability
- Time-Optimal Ergodic Control
- Energy-Optimal Ergodic Control

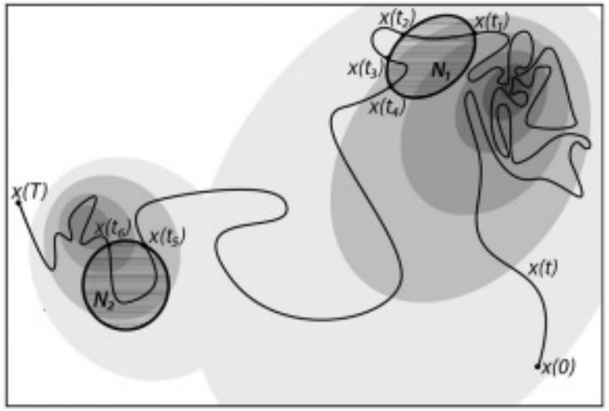
Introduction to Ergodicity

Ergodicity

- A trajectory is said to be *ergodic* if, on average, it spends time in regions proportional to the utility of exploring said area
- For $t_f \rightarrow \infty$, ergodic trajectories guarantee complete coverage

$$\lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} \phi(g \circ x(t))dt = \int_{\mathcal{W}} \phi(w)\mu(w)dw$$





[1] Ayvali, Elif, Hadi Salman, and Howie Choset. "Ergodic coverage in constrained environments using stochastic trajectory optimization." 2017 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS). IEEE, 2017.
[2] Miller, Lauren M., et al. "Ergodic exploration of distributed information." IEEE Transactions on Robotics 32.1 (2015): 36-52.

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Safety Guarantees in Ergodic Control (CBF)

Problem Formulation

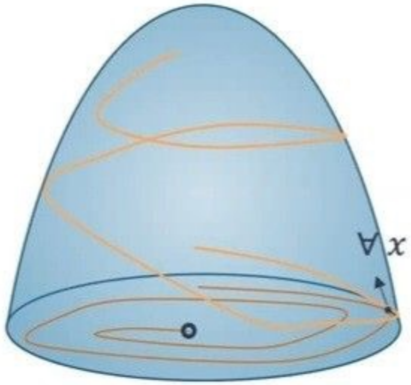
- Given an area with many obstacles, guarantee safe navigation without loss of coverage quality
- Safety defined as guaranteeing *set-invariance* over safe-set (using a Control Barrier Function[1])

$$\mathcal{C} = \{x \in \mathcal{X} \mid B(x) \geq 0\}$$

$$\min_{u} \|u - u_{\text{nom}}\|$$

$$\text{s.t.}\;\; \dot{B}(x,u) = \nabla B(x) \cdot f(x,u) \geq -\gamma(B(x)) \;\; \forall u \in \mathcal{U}$$

$$\text{and}\;\; \dot{x} = f(x,u)$$



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[1] Ames, Aaron D., Samuel Coogan, Magnus Egerstedt, Gennaro Notomista, Koushil Sreenath, and Paulo Tabuada. "Control barrier functions: Theory and applications." In *2019 18th European control conference (ECC)*, pp. 3420-3431. IEEE, 2019.

Safety Guarantees in Ergodic Control (CBF)

Problem Formulation

Issue:

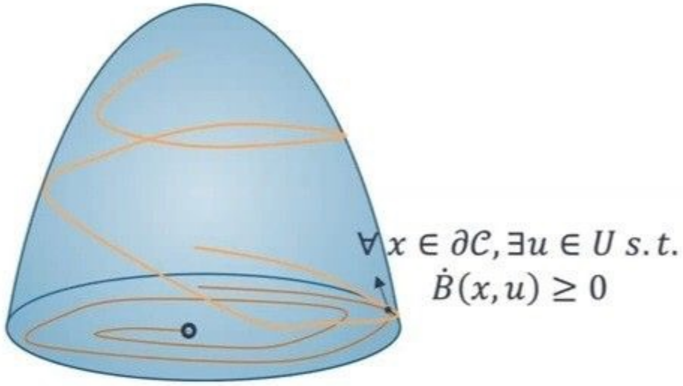
- Safety “filter” overrides future actions that inhibit effective exploration

$$\mathcal{C} = \{x \in \mathcal{X} \mid B(x) \geq 0\}$$

$$\min_{u} \|u - u_{\text{nom}}\|$$

$$\text{s.t. } \dot{B}(x,u) = \nabla B(x) \cdot f(x,u) \geq -\gamma(B(x)) \;\; \forall u \in \mathcal{U}$$

$$\text{and } \dot{x} = f(x,u)$$



Safety Guarantees in Ergodic Control (CBF)

Solution

- Jointly solve for coverage + safety
  - We solve problem over discrete trajectories

$$\min_{\mathbf{x},\mathbf{u}} \mathcal{E}(\mathbf{x},\phi) + \sum_{0}^{T-1} u_t^\top R u_t$$

$$\text{s.t.} \quad \begin{cases} x_{t+1} = f(x_t, u_t), x_t \in \mathcal{X}, u_t \in \mathcal{U} \\ x_0 = \bar{x}_0, x_{T-1} = \bar{x}_f, g(x) \in \mathcal{W} \\ \Delta B(x_t, u_t) \geq -\gamma B(x_t) \end{cases}$$

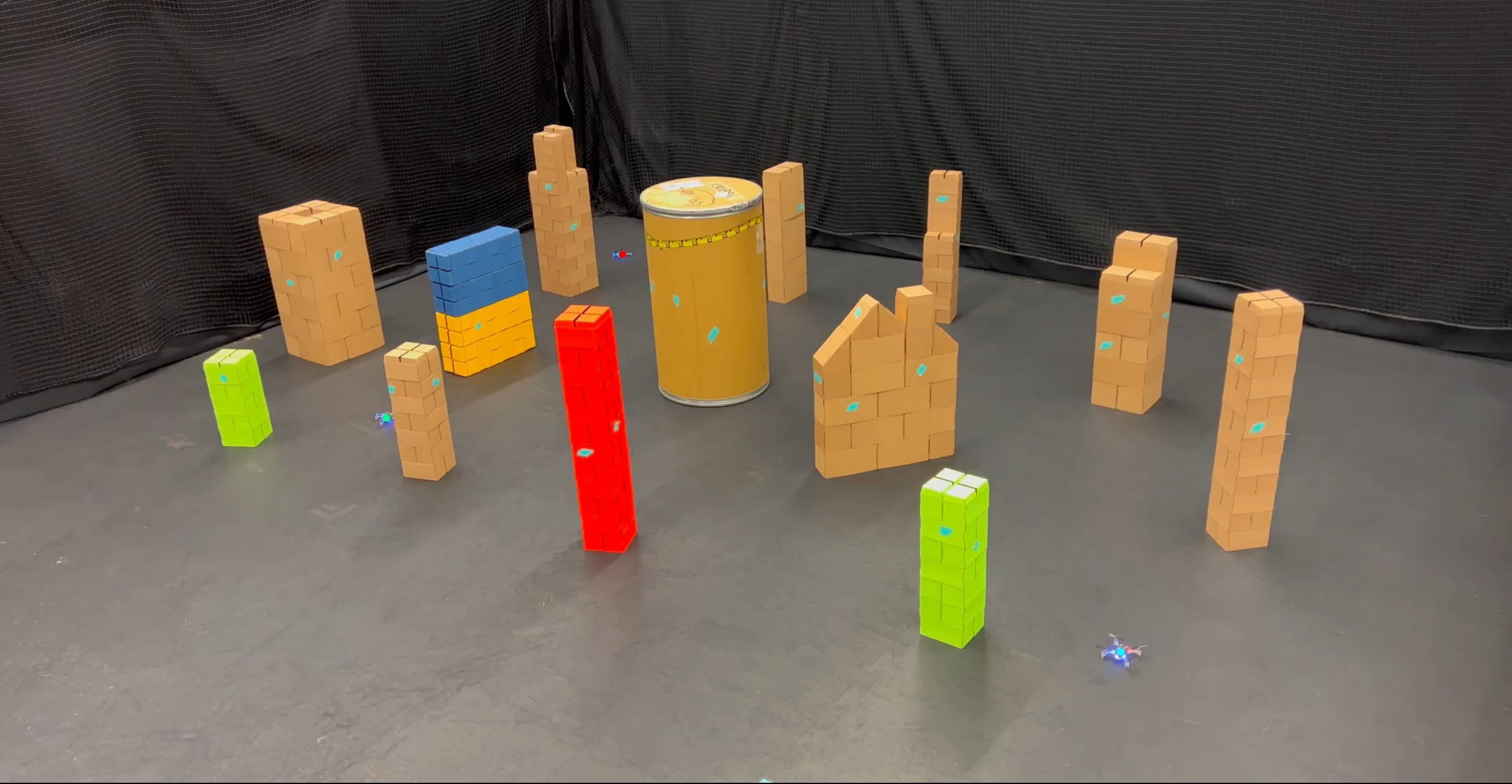
$$\mathcal{E}(\mathbf{x},\phi) = \sum_{k\in\mathbb{N}^v} \Lambda_k \left(c_k(\mathbf{x}) - \phi_k\right)^2$$

$$= \sum_{k\in\mathbb{N}^v} \Lambda_k \left(\frac{1}{T}\sum_{t=0}^{T-1} F_k(g(x_t)) - \int_{\mathcal{W}} \phi(w) F_k(w) dw\right)^2$$

Safety Guarantees

- Satisfied through barrier constraint
- Ergodic metric local minima provide additional flexibility
- Generic to various notions of safety

[1] Lerch, Cameron, Dayi Dong, and Ian Abraham. "Safety-critical ergodic exploration in cluttered environments via control barrier functions." In 2023 IEEE International Conference on Robotics and Automation (ICRA), pp. 10205-10211. IEEE, 2023.



Multi-drone ergodic search with drone-drone and drone-obstacle avoidance using

- Safety Guarantees in Ergodic Control
  - Set-Invariance with Control Barrier Functions
  - Hamilton Jacobi Isaacs Reachability
- Time-Optimal Ergodic Control
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Reachability in Ergodic Control

Problem Formulation

- Can we guarantee coverage fidelity under dynamic disturbances?
- Problem can be thought of as a *reachability problem* on achieving levels of *ergodicity*

Hamilton-Jacobi-Isaacs Reachability Problem

Find $V : \mathcal{X} \times [t_0, t_f] \to \mathbb{R}$ that satisfies

$$\frac{\partial V}{\partial t} = -\min_{u(t)} \max_{d(t)} \left\{ H(x, u, d, t) \right\}$$

$$\text{and } V(x, t_f) = m(x),$$

where the Hamiltonian H is defined as

$$H(x, u, d, t) \triangleq \ell(x, u, d, t) + \left(\frac{\partial V}{\partial x} \right)^\top f(x, u, d, t).$$

$$\min_{x(t), u(t)} \left\{ \mathcal{E}(x(t), \phi) + \int_{t_0}^{t_f} u(t)^\top \mathbf{R} u(t) dt \right\}$$

$$\text{s.t.} \quad \begin{cases} x \in \mathcal{X}, u \in \mathcal{U}, g(x) \in \mathcal{W} \\ x(t_0) = \bar{x}_0, x(t_f) = \bar{x}_f \\ \dot{x} = f(x, u) \\ h_1(x, u) \leq 0, h_2(x, u) = 0 \end{cases}$$

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Reachability in Ergodic Control

Extended Ergodic State

- We can define *an extended state* that converts problem into Bolza-form for HJI

Definition 1 ***Extended ergodic state.*** *The ergodic metric can be equivalently expressed as*

$$\mathcal{E}(x(t),\phi,t_f) = \sum_{k\in\mathcal{K}^v} \Lambda_k \left(c_k(x(t),t_f) - \phi_k\right)^2 = \frac{1}{t_f^2}\|z(t_f)\|^2_{\mathbf{\Lambda}}$$

where $z(t_f) = [z_0, z_1, \dots, z_{|\mathcal{K}^v|}]^\top$ *is the solution to*

$$\dot{z}_k = F_k(g(x(t))) - \phi_k \tag{1}$$

with initial condition $z(t_0) = \mathbf{0}$*, and* $\mathbf{\Lambda} = diag(\Lambda)$ *is a diagonal matrix consisting of the weights* $\Lambda = [\Lambda_0, \dots, \Lambda_{|\mathcal{K}^v|}]$.

Extended Ergodic Control Problem w/ Disturbance

Find $u : [t_0, t_f] \to \mathcal{U}$ that minimizes

$$\mathcal{J}_{\text{worst case}} = \max_d \left\{ \frac{1}{t_f^2}\|z(t_f)\|^2_\Lambda + \int_{t_0}^{t_f} \ell(x(\tau),u(\tau))d\tau \right\},$$

subject to

$$\dot{x}(t) = f(x(t),u(t),d(t))$$
$$\dot{z}_k(t) = F_k(g(x(t))) - \phi_k, \forall k \in \mathcal{K}$$
$$\text{and } z_k(t_0) = 0$$

[1] De La Torre, Gerardo, et al. "Ergodic exploration with stochastic sensor dynamics." *2016 American Control Conference (ACC)*. IEEE, 2016.
[2] Mathew, George, and Igor Mezić. "Metrics for ergodicity and design of ergodic dynamics for multi-agent systems." *Physica D: Nonlinear Phenomena* 240.4-5 (2011): 432-442.

Reachability in Ergodic Control

Extended Ergodic State

- We can define *an extended state* that converts problem into Bolza-form for HJI

Extended Ergodic Control Problem w/ Disturbance

Find $u : [t_0, t_f] \to \mathcal{U}$ that minimizes

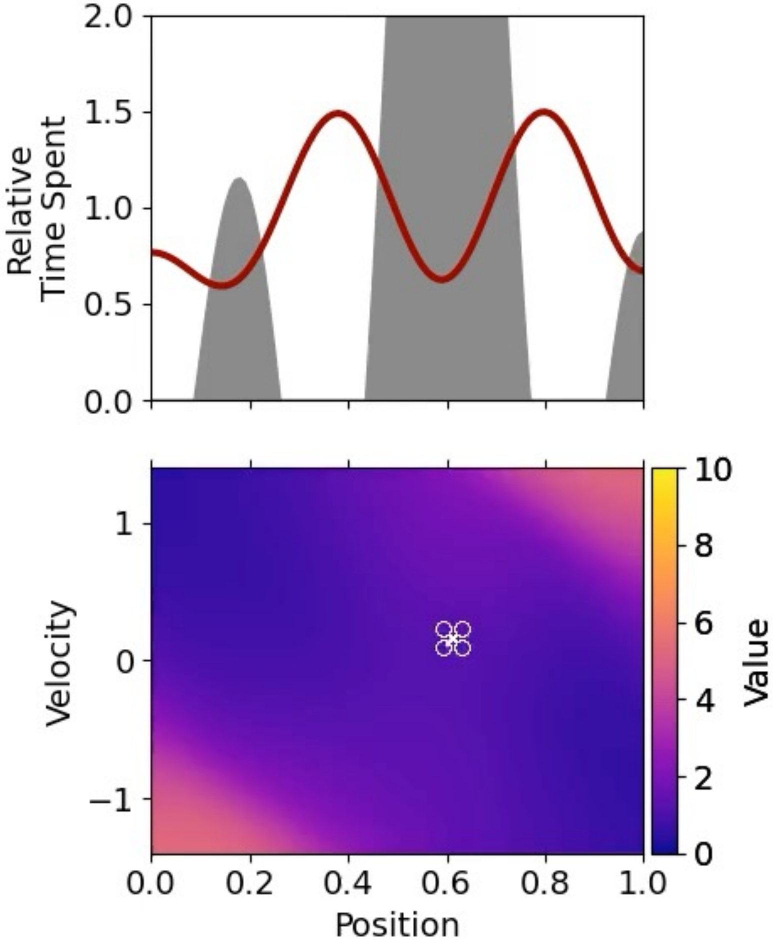
$$\mathcal{J}_{\text{worst case}} = \max_{d} \left\{ \frac{1}{t_f^2} \|z(t_f)\|_{\Lambda}^2 + \int_{t_0}^{t_f} \ell(x(\tau), u(\tau)) d\tau \right\}, \tag{1a}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), d(t)) \tag{1b}$$

$$\dot{z}_k(t) = F_k(g(x(t))) - \phi_k, \forall k \in \mathcal{K} \tag{1c}$$

$$\text{and } z_k(t_0) = 0 \tag{1d}$$



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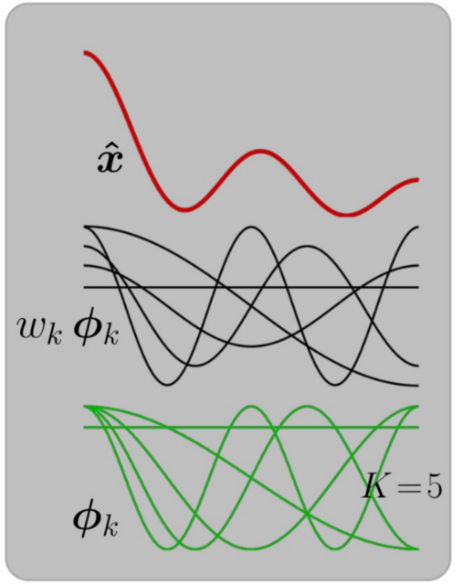
Time-Optimality in Ergodic Control



[1] Calinon, 2020

Problem Formulation

- What is the minimum time a robot needs to explore an area?
  - Ergodic metric based on average time trajectory visits area
  - Metric calculated using Fourier spectral decomposition
    - promotes uniform coverage based on utility measure

$$\mathcal{E}(x(t),\phi) = \sum_{k\in\mathcal{K}^v} \Lambda_k \left(c_k - \phi_k\right)^2$$

$$= \sum_{k\in\mathcal{K}^v} \Lambda_k \left(\frac{1}{t_f}\int_{t_0}^{t_f} F_k(g(x(t)))dt - \int_{\mathcal{W}} \phi(w)F_k(w)dw\right)^2$$

  - Ergodic metric is inversely proportional to time and can act as constraint to bound time

[1] Calinon, Sylvain. "Mixture models for the analysis, edition, and synthesis of continuous time series." *Mixture Models and Applications* (2020): 39-57.
[2] Dong, Dayi, Henry Berger, and Ian Abraham. "Time Optimal Ergodic Search." Robotics: Science and Systems (2023).

Time-Optimality in Ergodic Control

Problem Formulation

- We can rewrite ergodic trajectory optimization as a *minimum-time problem* with the *ergodic metric as a constraint*!

Time-Optimal Ergodic Trajectory Optimization:

$$\boxed{\min_{x(t),u(t),t_f} \quad t_f}$$

$$\text{s.t.} \quad \begin{cases} x \in \mathcal{X}, u \in \mathcal{U} \\ x(t_0) = \bar{x}_0 \\ \dot{x} = f(x,u) \\ x(t_f) = \bar{x}_f, g(x) \in \mathcal{W} \\ h_1(x,u) \leq 0, h_2(x,u) = 0 \\ \mathcal{E}(x(t), \phi, t_f) \leq \gamma, t_f > 0 \end{cases}$$

Minimize time

Subject to dynamics, equality, and inequality constraints,

And minimum ergodic coverage constraint w.r.t. utility measure

Time-Optimality in Ergodic Control

Numerical Solutions

- Locally optimal conditions proven to exist (based on Maximum Principle)
  - Computationally hard to compute
- Direct transcription can be used to compute solutions (based on KKT conditions)

$$
\min_{\mathbf{x},\mathbf{u},t_f} \quad t_f
$$

$$
\text{s.t.} \quad \begin{cases} \boxed{\Delta t = \frac{t_f}{N}} \\ x_t \in \mathcal{X}, u_t \in \mathcal{U} \\ x_0 = x_0, \boxed{x_{t+1} = x_t + \Delta t f(x_t, u_t)} \\ x_N = x_f, g(x_t) \in \mathcal{W} \\ h_1(\mathbf{x}, \mathbf{u}) \leq 0, h_2(\mathbf{x}, \mathbf{u}) = 0 \\ \boxed{\hat{\mathcal{E}}(\mathbf{x}, \phi, t_f) \leq \gamma}, t_f > 0 \end{cases}
$$

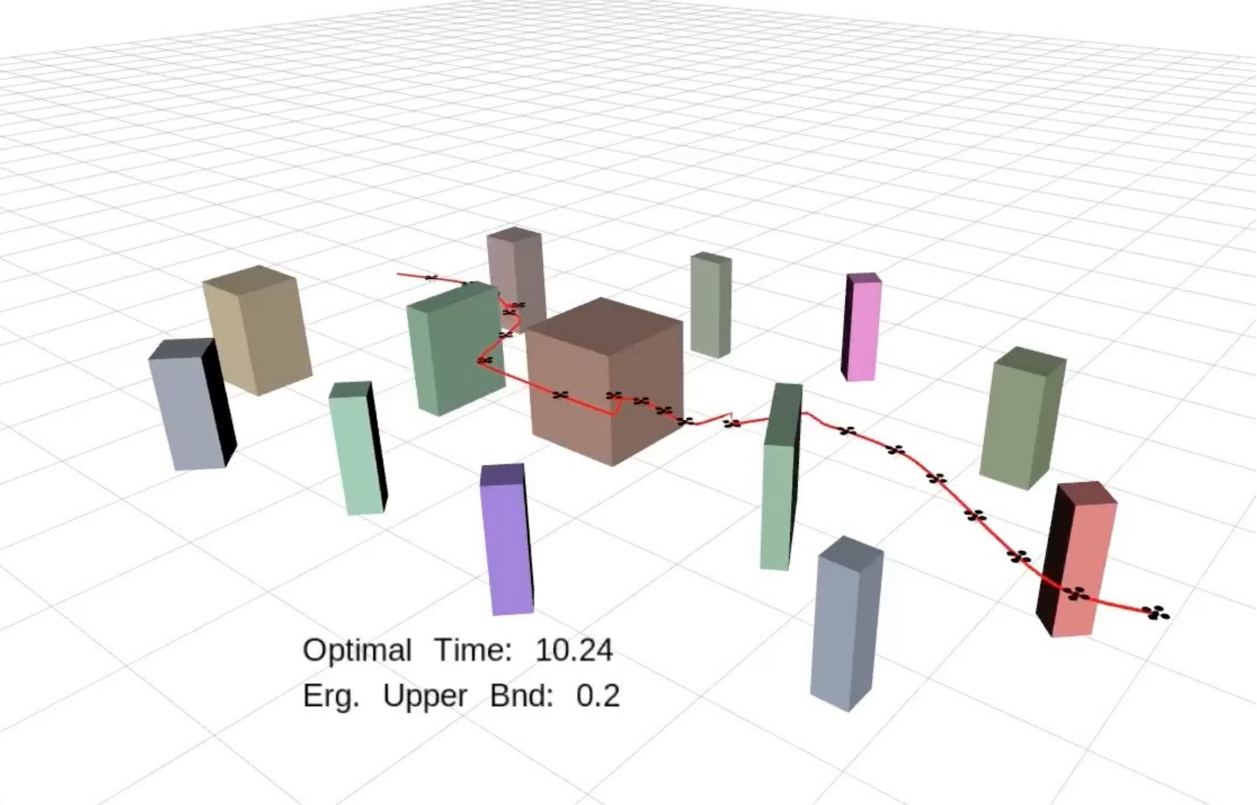
Step-size encoded through N knot points

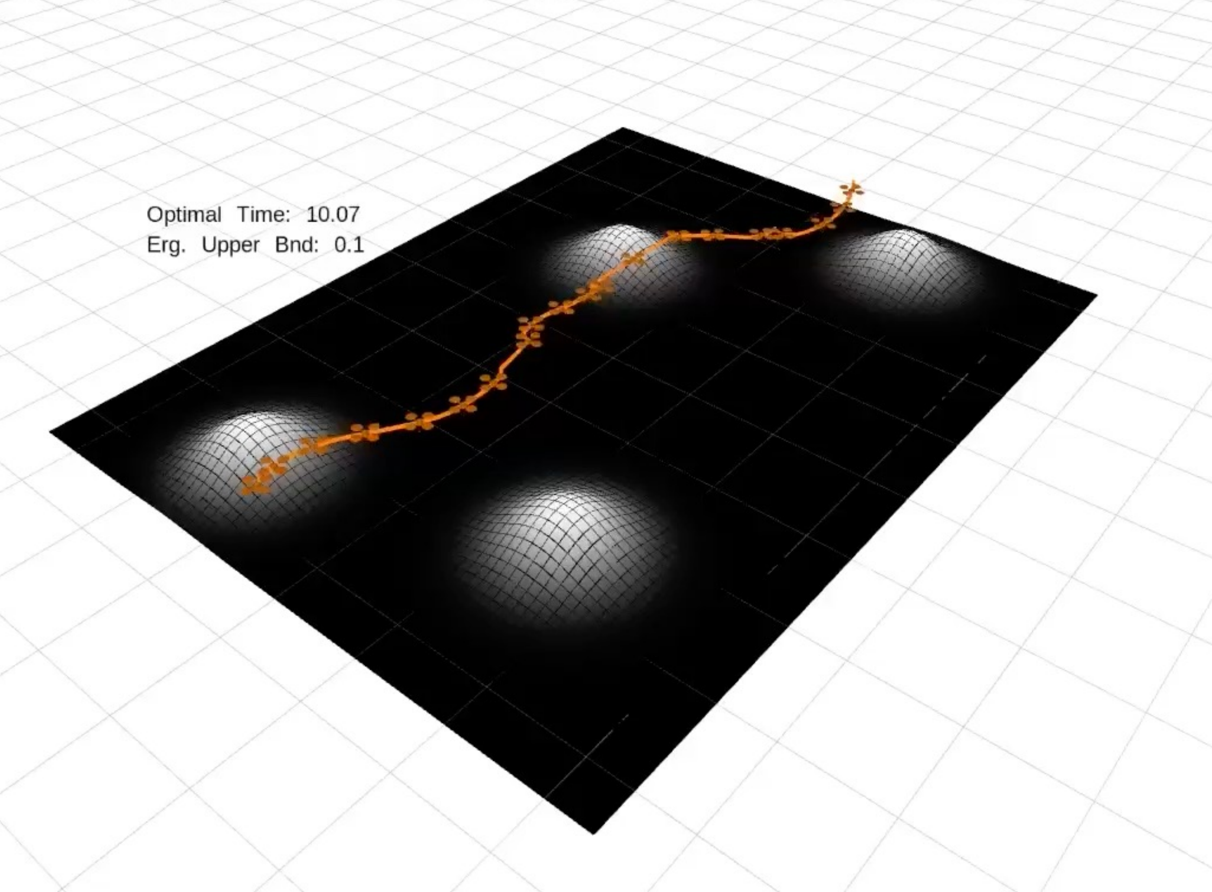
Trajectory transcribed over N knot points

Ergodic metric calculated over discrete-time trajectory

[1] Dong, Dayi, Henry Berger, and Ian Abraham. "Time Optimal Ergodic Search." Robotics: Science and Systems (2023).

Example Time-Optimal Ergodic Trajectories





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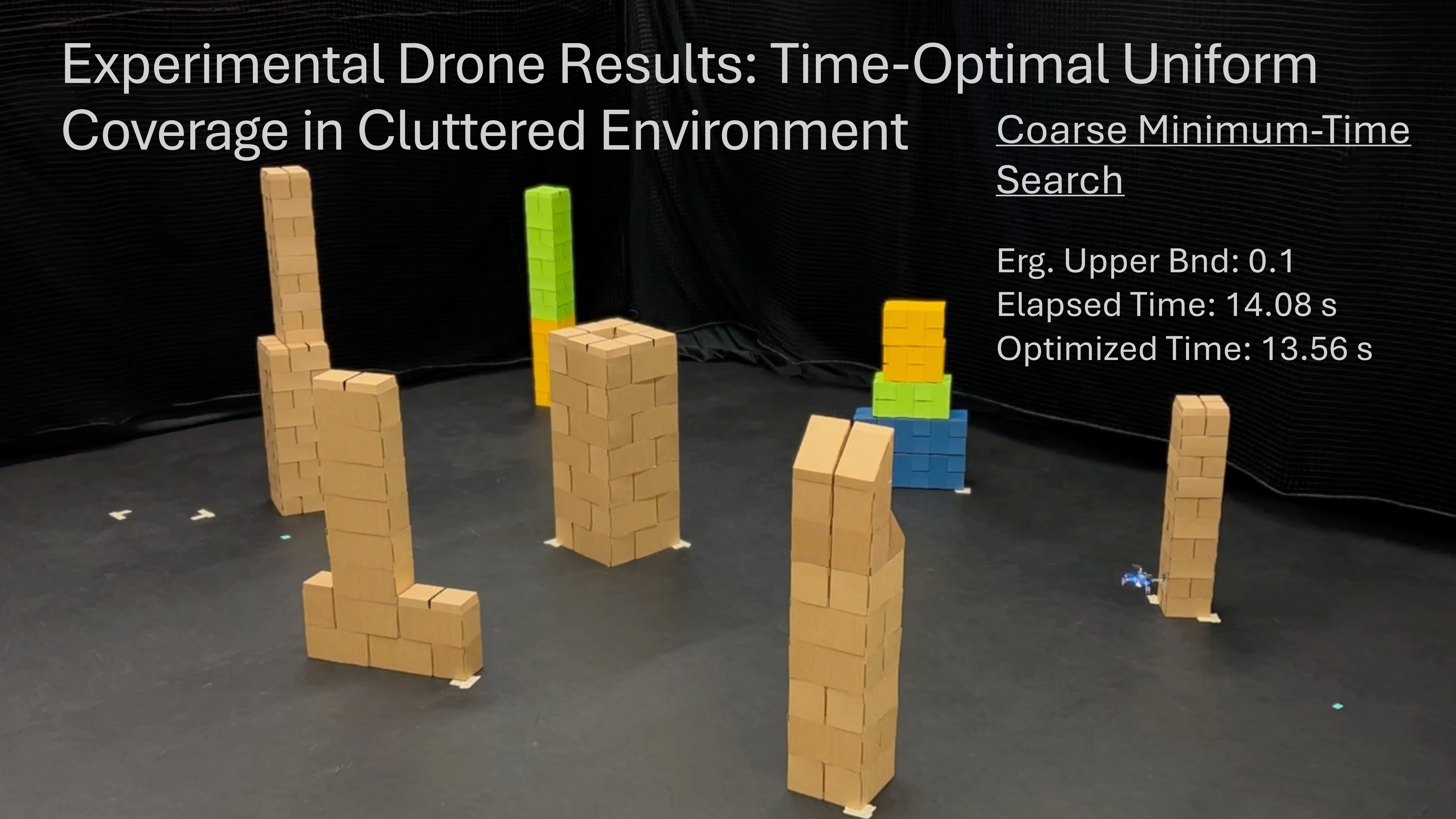
[1] Dong, Dayi, Henry Berger, and Ian Abraham. "Time Optimal Ergodic Search." Robotics: Science and Systems (2023).

Experimental Drone Results: Time-Optimal Uniform Coverage in Cluttered Environment



Coarse Minimum-Time Search

Erg. Upper Bnd: 0.1
Elapsed Time: 14.08 s
Optimized Time: 13.56 s